Curriculum Branch

SENIOR HIGH MATHEMATICS 30/33

Interim Teacher Resource Manual

1991

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MATHEMATICS 30/33

INTERIM TEACHER RESOURCE MANUAL

Curriculum Branch
Field Test

DRAFT, April 1991

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ACKNOWLEDGMENTS

Alberta Education expresses its appreciation to those individuals and groups who assisted in the development and production of this manual and acknowledges the contribution of the following:

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January 7, 1992

Ms Lin Mireau
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University of Alberta
Edmonton, Alberta
T6G 2G5

Dear Ms Mireau:

As you requested, I am enclosing fifteen copies of the Mathematics 30-33 Teacher Resource Manuals (Interim). In addition, I have enclosed 15 copies of the Quadratic Relations supplement that replaces the Quadratic Relations unit in the TRM.

We are currently reviewing the three high school TRM's with a view to publishing a single version which would contain materials for Math 10-20-30, 13-23-33, 14-24 and eventually 31.

Sincerely,

Hugh Sanders

Hugh Sanders
Program Manager
Mathematics
Curriculum Branch

HS/tp

Enclosures

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PURPOSE

This Teacher Resource Manual (TRM) is designed to help teachers implement the Mathematics 30, 33 courses. The TRM is a support document that provides helpful information to classroom teachers. The teaching strategies presented are suggestions only and are not legally binding. The courses of study for Mathematics 30 and 33 outline the material that teachers are legally bound to teach. An overview of the courses of study for each of Mathematics 30 and 33 is provided at the beginning of each course section in the TRM. The overview includes the specific learner expectations, the attitudinal expectations and the problem-solving expectations for each course.

The remaining portions of each Course of Studies: Program Rationale and Philosophy, General Learner Expectations and Specific Learner Expectations, are stated at the beginning of the document and shaded.

The Mathematics 30/33 TRM is intended to help teachers translate the Senior High Mathematics Program into classroom practice.

PROGRAM RATIONALE AND PHILOSOPHY

To set goals and make informed choices, students need an array of thinking and problem-solving skills. Fundamental to this is an understanding of mathematical techniques and processes that will enable them to apply the basic skills necessary to address everyday mathematical situations, as well as acquire higher order skills in logical analysis and methods for making valid inferences.

A knowledge of mathematics is essential for a well-educated citizenry. However, the need for and use of mathematics in the life of the average citizen is changing. Emphasis has shifted from the memorization of mathematical formulas and algorithms toward a more dynamic view of mathematics as a precise language, used to reason, interpret and explore. There continues to be a need for the logical development of concepts and skills as a basis for the appropriate use of mathematical information to solve problems. Moreover, the use of available technology along with techniques such as estimation and simulation, incorporated with more traditional problem-solving techniques, are the tools by which mathematical problems are solved.

Change in the way in which mathematics is used is necessitating a concurrent change in the emphases of mathematics education. Students need an expanded list of fundamental concepts but will also need to understand the ideas that make up those concepts and how they are related. They also require a familiarity with their applications. Most important, students have to be able to solve problems using the mathematical processes developed, and be confident in their ability to apply known mathematical skills and concepts in the acquisition of new mathematical knowledge. In addition, the ability of technology to provide quick and accurate computation and manipulation, to enhance conceptual understanding and to facilitate higher order thinking, should be recognized and used by students.

The majority of students who enter high school exhibit mainly concrete operational behaviours with regard to mathematics. It is recognized that high school mathematics courses include many abstract understandings that students are expected to acquire. The course content of the high school mathematics program is cognitively appropriate for the students and should be presented in a way that is consistent with the students' ability to understand.

The Senior High School Mathematics Program includes three course sequences, which are Mathematics 14/24, 13/23/33 and 10/20/30. Transfer by students among courses of different sequences is possible. The course sequences commensurate with differing abilities, interests and aspirations, are designed to enable students to have success in mathematics. As well, the mathematics program reflects the changing needs of society, and provides students with the mathematical concepts, skills and attitudes necessary to cope with the challenges of the future.

GENERAL LEARNER EXPECTATIONS

Upon conclusion of their senior high school mathematics education, students will be expected to be mathematically literate. **Mathematical literacy** refers to students' ability and inclination to manage the demands of their world through the use of mathematical concepts and procedures to communicate, reason and solve problems. More specifically, *students will be expected to:*

- have achieved understanding of the basic mathematical concepts, and developed the skills and attitudes needed to become responsible and contributing members of society
- apply basic mathematical skills and concepts in practical situations
- have developed the skills, concepts and attitudes that will ensure success in the mathematical situations that occur in future educational endeavours, employment and everyday life
- have developed the skills, concepts and attitudes that will enable the acquisition of mathematical knowledge beyond the conclusion of secondary education
- have developed critical and creative thinking skills
- be able to communicate mathematical ideas effectively
- understand how mathematics can be used to investigate, interpret and make decisions in human affairs
- understand how mathematics can be used in the analysis of natural phenomena
- understand the connections and interplay among various mathematical concepts and between mathematics and other disciplines
- understand and appreciate the positive contributions of mathematics, as a science and as an art, to civilization and culture.

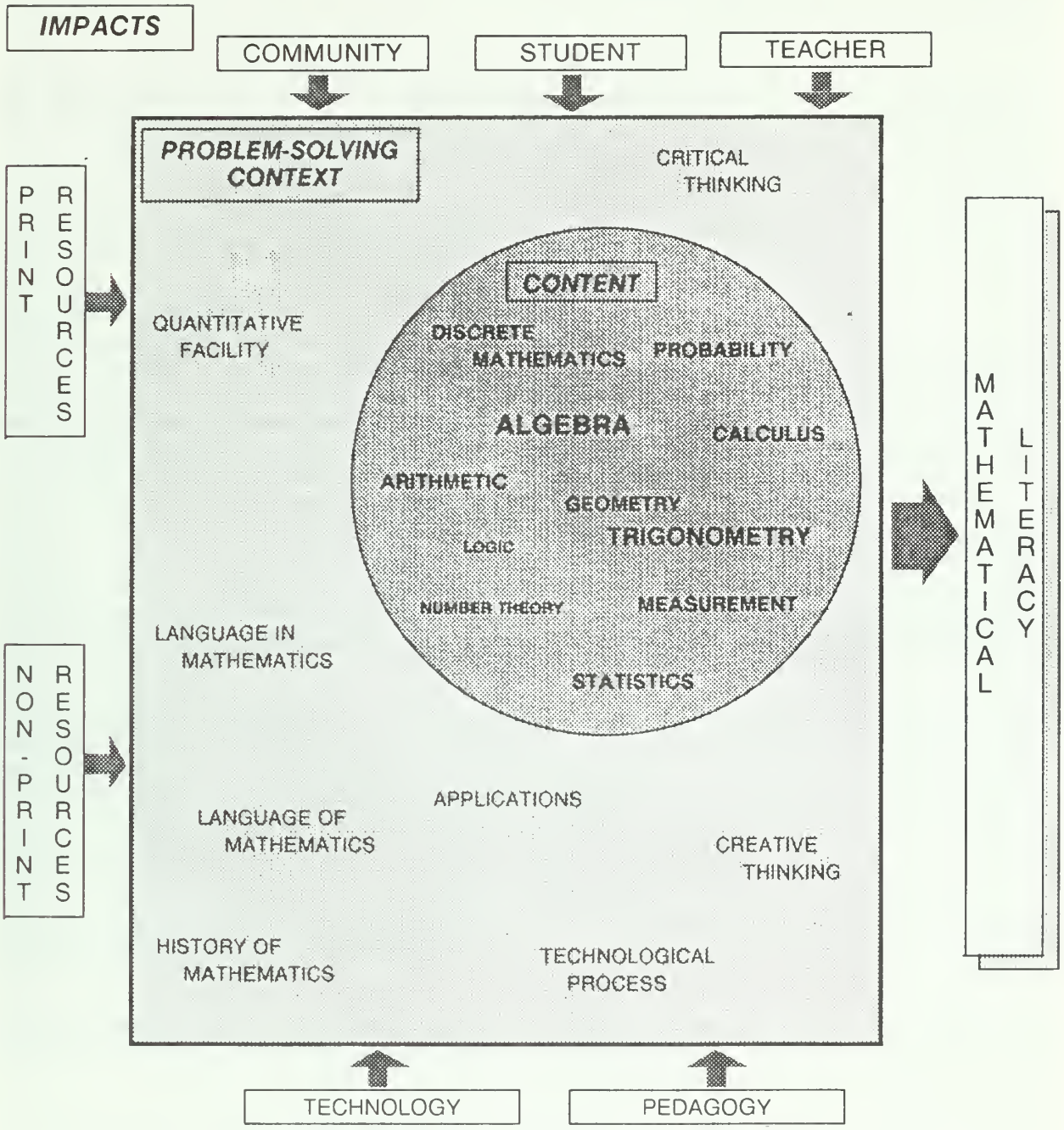
A General Model for **Mathematical Literacy** for Senior High School Programs is presented on the following page, which outlines the factors that affect what and how students learn as they become mathematically literate. The model is fluid in that the content can be learned within a problem-solving context that engages any of a number of vehicles as the learning focus. At the same time, the students involved in the learning situation are affected in what and how they learn by forces that have impact on them.

The **Content** of the Senior High School Mathematics Program is the body of knowledge that is to be acquired by students. In the various senior high school mathematics courses, it is made up of topics that can be categorized into one or more of the listed strands.

Within each course in the Senior High School Mathematics Program, students will focus on problem solving. The **Problem-Solving Context** refers to the instructional emphases within which the specific content expectations can be acquired. The various entries indicated within the model suggest processes that belong to the problem-solving context and may be used by students as vehicles for learning the content.

The **Impacts** on the problem-solving context are those skills, attitudes and experiences that are possessed by the students and teachers involved, as well as the resources they may use throughout the learning process. They include the influence exerted by the culture and beliefs of the community as reflected by the school. The effectiveness of the context in enabling a student to acquire the content is dependent on the skilful management by the teacher of those items that have an impact on a student's learning.

A General Model for Mathematical Literacy for Senior High School Programs



SPECIFIC LEARNER EXPECTATIONS

PROGRAM ORGANIZATION

The major part of the content of each senior high school mathematics course consists of topics required of all students who take the course. The required content comprises 80% of the course and contains the concepts, skills and attitudes that all students are expected to acquire. As well, the required portion of all courses includes specific expectations for problem solving and the use of technology.

Each course includes a compulsory component comprising 20% of the course, made up of elective material that is consistent with the content and expectations of the required component. The purpose of the elective material is to provide for enrichment, remediation, or innovative or experimental presentations or activities. It is not intended to provide acceleration or advanced placement. However, horizontal enrichment and extension is appropriate and students should have access to elective material that serves their individual needs and interests.

Evaluation of students in the Senior High School Mathematics Program will involve assessment of the level of achievement of all of the learner expectations, including concepts, skills and attitudes, as well as problem solving and technology expectations. For more information regarding evaluation, consult the *Teacher Resource Manual for Senior High Mathematics.*

PROGRAM STRUCTURE

At the beginning of each course is a list of attitude expectations. These attitudes embody a mathematical attitude or frame of mind for a student to view the world. The attitude expectations should be woven into the fabric of the entire course.

Following this are the problem-solving expectations that outline a variety of procedures, strategies, skills and checking techniques for solving problems. Because a major purpose for studying mathematics is to learn to solve problems, problem-solving expectations occur throughout all areas of the specific learner expectations. Students must have the background skills and knowledge necessary to achieve these expectations successfully, using problem-solving techniques.

The units of the course are broken into a number of concepts. A concept is an abstract or general idea about specific instances that have common properties or an identifiable relationship to one another. The concepts are presented as mathematical definitions or theorems or as statements of mathematical ideas or abstractions. In the course of studies, concepts are stated as "*Students will be expected* to demonstrate an understanding that . . ."

Supporting each concept are a number of skills. Skills are intellectual or physical capabilities that will be developed within the context of the particular concept. In the course of studies, skills are stated as "*Students will be expected to . . .*"

Skills specifically related to the use of technology identify areas in which calculators and/or computer technology are applied by students as tools to be used for calculations, manipulation or graphing or to aid in the analysis of problems. Technology expectations are defined explicitly throughout the learner expectations. In many cases, a particular technology is indicated for investigation or analysis. It is in these situations that the use of technology enables students to engage in critical and creative thinking and problem solving.

Students will be expected to learn how and when to use a calculator and have a demonstrated proficiency in estimation and mental arithmetic. To use calculators effectively, they must be able to judge the reasonableness of an answer and understand the importance of making a judgment about the result of a calculation.

Throughout the learner expectations, the words verify and prove appear. For the purposes of the Senior High Mathematics Program, they are interpreted as:

- Verify: to substantiate the validity of an operation, solution, formula or theorem through the use of examples that may or may not be generalized;
- Prove: to substantiate the validity of an operation, solution, formula or theorem in general and to provide logical arguments for each step in the process.

PROBLEM SOLVING

Through the use of problems and applications, students will be able to see the need for certain concepts and mathematical procedures, to understand concepts and acquire facility with procedures, and to apply and review concepts and procedures already learned. This emphasis on problem solving in high school mathematics has shifted from finding answers to routine "word problems," to the application of mathematical concepts, skills and processes to develop and discover mathematics within a wide variety of problem situations.

Problem solving has been integrated into all of the courses in the Senior High Mathematics Program and is explicitly mentioned throughout the learner expectations. In addition, specific expectations for problem solving are included at the beginning of each course. It is not intended that problem solving be presented as units of study; however, it is expected that upon completion of the course, students will have acquired all of the listed skills and concepts.

Problem solving is the focus of the concept development as well as the exercise portion, of a lesson. New material should be presented as a problem to be solved using previously acquired skills, concepts and processes. Teachers must be open to accepting non-rigorous strategies such as modelling, working backward and simulations, and should emphasize the processes used and the type of thinking in which the student is engaged at least as much as the answer to the problem. Teachers must remember that what may be a problem to some students may be only a routine exercise to others, and should be prepared to adapt their presentations and expectations to reflect this. Above all, teachers must model problem-solving behaviour; students will think problem solving is important if they are encouraged by their teachers to solve problems, and if they see their teachers solving problems.

At all levels of problem solving, students should be taught how and provided with the opportunities to ask questions and to analyse the thinking processes with which they tackle a problem. An exercise that provides students with this opportunity is outlined below.

"Archie's Envelopes"
Students work in three situations:

1. A group of four students is involved in solving a problem. Each student is given only one piece of information from the problem, in an envelope. Each student must explain their piece of information to the group. The group must:
   a) identify "the problem";
   b) plan a solution;

c) try out the plan; and
d) analyse the "plan" and method of solution.

2. Same as situation 1, with two students.
3. Students work individually on solving a problem.

(Thanks to A. Coderre, Math 10 Field Test Teacher, for this idea.)

As students become more adept at problem solving, they will begin to use more complex problem-solving processes. Such higher level processes will include making and testing conjectures, determining valid inferences, using inductive and deductive processes, and developing and using algorithms to apply to a particular situation.

Through the course of studies, verbs such as "verify" and "prove" are used. These expectations suggest explicitly to the teacher that a student should have the background skills and knowledge necessary to achieve the expectation successfully using problem-solving techniques. Teachers are encouraged to adhere to the spirit of these expectations by treating them as problems to be solved. Students are more likely to internalize the skills or concepts if they are encouraged to develop them themselves.

TECHNOLOGY

Students will be expected to learn how and when to use a calculator and have a demonstrated proficiency in estimation and mental arithmetic.

To use calculators effectively, they must be able to judge the reasonableness of an answer and understand the importance of making a judgment about the result of a calculation.

Throughout the learner expectations, specific applications of calculators and computers are identified. These define explicitly the areas in which students are expected to use technology in their study of mathematics. Many of these expectations indicate that a particular technology be used in an investigation or for analysis. It is in these situations that the use of technology enables students to engage in critical and creative thinking and problem solving.

The integration of calculators into all courses in the high school mathematics program – in support of instruction, in class work, in homework and evaluation – is essential. By using calculators appropriately, time previously spent on computation can be used for problem solving.

The use of computers by teachers and students in mathematics classes should be expanded. Teaching using computers allows concepts to be presented visually through graphics that can be manipulated by the teacher according to input supplied by the students. Students should be taught to use prepared software and to use simple programs written for particular purposes, to use the power of the computer to create geometric displays, organize, analyse and present data, simulate real-life situations, and create and run algorithms, as well as to do symbolic manipulations and calculations.

Electronic technology will continue to play an important role in the teaching of mathematics in both what is taught and how it is taught. Teachers should encourage students to use available technology as much and as well as possible and should adapt their teaching techniques and evaluation emphases to reflect this.

PLANNING FOR INSTRUCTION

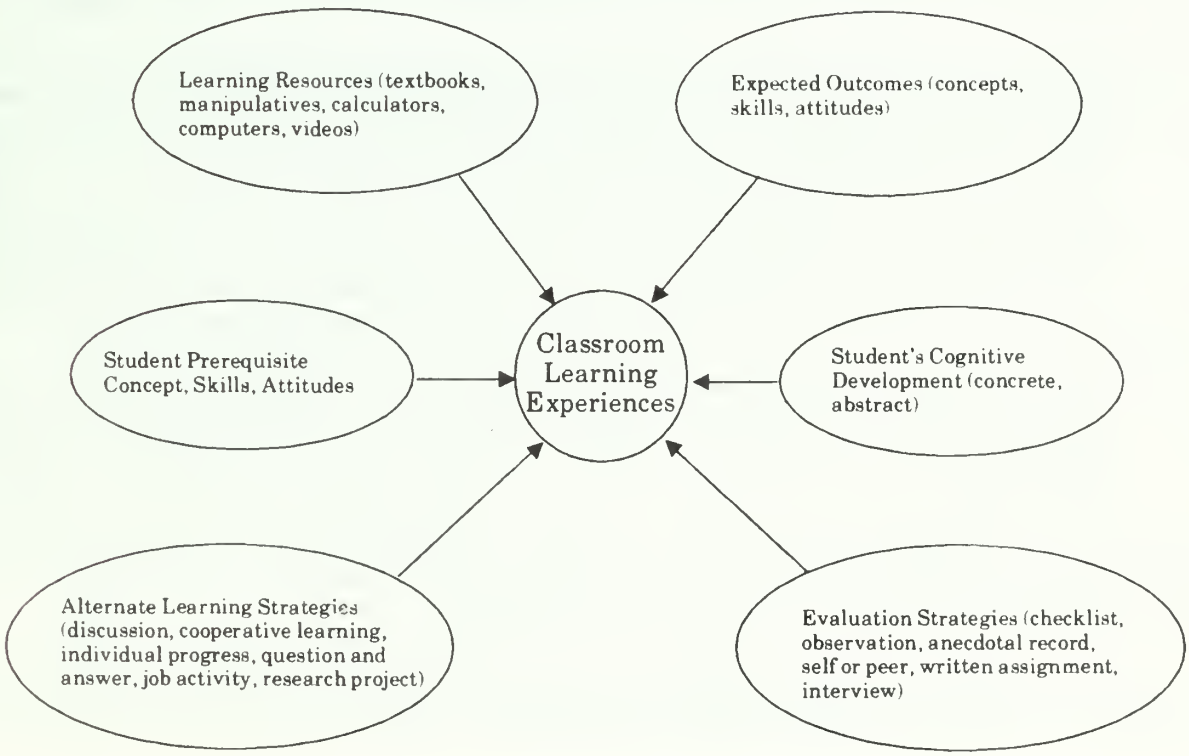
When planning, teachers should begin at the end by identifying what students should know, be able to do, and feel after instruction has occurred. Planning involves determining the destination that teachers want their students to reach and plotting the manner in which the teacher will help students reach that destination. The basic elements of planning are to:

- identify where the students are going (general and specific learner expectations – concepts, skills and attitudes)
- identify how the students will get there (learning strategies)
- identify how you and the students will know when the students have arrived (evaluation strategies).

Planning is a systematic approach to integrating learning experiences to achieve the concept, skill and attitude expectations. However, the instructional plan is a proposal, not a recipe. It should have sufficient flexibility to allow the teacher to relate to students' abilities and to take advantage of learning opportunities that may arise during instruction. The teacher should keep in mind that there are many ways to reach the same destination and not all students require the same learning strategy.

Instructional planning includes yearly, unit and daily lesson planning. The elements used in planning at the three levels are similar but vary in the degree of detail. The elements to consider in determining the instructional experiences that the student will receive include: the learner expectations, where the student is relative to these expectations, the cognitive ability of the student, the basic and support learning resources that will be used, alternative learning strategies and how the experience will be evaluated.

Elements of Planning


Learning Resources (textbooks, manipulatives, calculators, computers, videos)
Expected Outcomes (concepts, skills, attitudes)
Student Prerequisite Concept, Skills, Attitudes
Classroom Learning Experiences
Student's Cognitive Development (concrete, abstract)
Alternate Learning Strategies (discussion, cooperative learning, individual progress, question and answer, job activity, research project)
Evaluation Strategies (checklist, observation, anecdotal record, self or peer, written assignment, interview)

When the teacher is determining the sequence in which units and topics are to be presented they should consider the way their students learn and the way the resources they have selected are organized. The way the concepts, skills and attitudes are organized in the Course of Studies is not necessarily the way that learning will occur.

Long-range course and unit plans should be communicated to students, parents and administrators. Outlines distributed to students should include evaluation criteria, to allow students to assess their own progress and avoid any surprises at report card time.

Learning is enhanced when the expected outcomes of the course match the learning experiences and the methods of evaluation. When students are clear about where they are going and can see they are making progress toward this endpoint, they become motivated to continue to work toward attainment of the desired outcomes.

STUDENT EVALUATION

CHARACTERISTICS OF EFFECTIVE EVALUATION OF STUDENT LEARNING IN MATHEMATICS

1. There should be a strong and obvious connection among the learner expectations, the learning experiences in the classroom and methods of evaluation.

   - The activity, test or assignment must evaluate the student's success in meeting the learner expectations. For example, if the activity evaluates only content recall when the emphasis in the course is on the development of the students' abilities to analyze, interpret and apply concepts, little information will have been produced regarding student mastery of what the student is expected to learn. Evaluation situations that require students to respond to isolated questions on specific learner expectations do not reflect a mathematics curriculum that is an integration of mathematical concepts and procedures in a coherent, meaningful structure.

   - When assessment instruments are aligned with the designed curriculum and the instructional practices in the classroom, a very powerful and consistent message is delivered to the student. What is evaluated and how it is evaluated tells the student very clearly what is important in the course. For example, if the learner expectations stress acquisition of higher order thinking skills, but the evaluation emphasizes only factual recall and the teacher lectures most of the time, students will be confused over what exactly it is they are expected to be learning but will likely conclude that learning facts is most important. Since students are expected to be able to use calculators, computers and manipulatives, evaluation situations must not only reflect the use of these items but the test items must be appropriate for use with these materials.

   - Learning mathematics extends beyond just learning concepts, procedures and their application. It also includes developing a disposition toward mathematics and seeing mathematics as a powerful way of looking at situations. Disposition refers not simply to attitude but extends to a tendency to think and act in positive ways. Students' mathematical dispositions are apparent in the way they approach tasks – whether with confidence, willing to explore alternatives, perseverance and interest – and their tendency to reflect on their own thinking. Information about student dispositions to mathematics is best collected through observation of students as they attempt to solve problems.

   - The amount of alignment depends on the extent that the curricular emphasis is reflected in the assessment's relative emphasis. For example, an assessment instrument that contains a

large number of questions that ask the student to factor a trinomial and contains few problem-solving questions is poorly aligned with a curriculum that stresses problem solving and reasoning.

2. A variety of evaluation approaches should be used to collect information about students' learning.

   - Learning is a complex process, occurring in many ways and at many levels. Assessment situations should demand different kinds of mathematical thinking as well as present the same mathematical concept or procedure in different contexts, formats and problem situations. Since the mathematics curriculum supports an inquiry approach to instruction, stressing problem solving as the medium through which mathematics is learned and applied, assessment should cover many aspects of a single concept, use procedures in many contexts and require students to show they have integrated their knowledge.

   - No single evaluation instrument or test can provide all the information teachers require to assess what and how their students are learning. The quality of judgments about student attainment of the learner expectations is related to the consistency of the results obtained from a number of different sources. When a student performs similarly on many different tasks, teachers can have confidence in their judgment of that student.

3. Student evaluation requires careful planning and the development of appropriate evaluation criteria.

   - Evaluation is an integral part of planning; it is not something to be considered after instruction. Evaluation should be addressed during all phases of planning and instruction. Evaluation is an integral part of the teaching process, not separate or adjunct to it.

   - Before instruction can begin on a unit of study (year's work, unit or daily lesson) the teacher must determine the degree to which students have mastered the prerequisite concepts, skills and attitudes.

   - During planning the teacher must determine what will be accepted as evidence that the expectations for the unit of study have been acquired by the students. Part of this planning is to determine what standard of achievement the teacher is willing to accept (see section on standards).

   - Upon reviewing the result of an evaluation the teacher may decide to alter instructional strategies.

   - A long-range plan for student evaluation should be developed and communicated to students, parents and administrators early in the school term. This plan should indicate how the students' report card and final grade will be determined including a description of the various components that will be included and the relative weightings of these various components.

4. The purpose of an assessment should determine the kind of evaluation situation that will be used.

   - The assessment of student performance serves many purposes – to identify areas of difficulty for individual students, to gather data for instructional planning, to assign grades or evaluate a program. The purpose should dictate the kinds of questions asked, the methods employed and the uses made of the resulting information.

   - The student, the teacher, the school administrator and the general public all want different kinds of information because they are seeking answers to different questions. For example,

the results from standardized tests of general mathematical achievement are not necessarily an appropriate measure of the effectiveness of a specific curriculum. Standardized achievement tests are designed to measure an individual student's relative position in a population and must maximize individual differences among students while measuring the common elements of their instruction. As such, these tests reflect a student's general achievement, background and prior knowledge and therefore may not measure the amount of learning that has occurred as the result of any one instructional intervention.

5. Evaluation should be continuous.

   - During instruction teachers are continually making judgments about student progress. As teachers are working with students they are making instructional decisions based on the feedback students are giving them. As teachers are talking they are scanning the students for signs of recognition, they ask questions to determine if students understand the material, they listen to students talk to see if students can verbalize what is being studied, they watch students work together on a problem. These informal observations often provide insights into the thoughts of students and can be a very rich source of information for important instructional decisions.

   - The approaches used to evaluate students' performance must take into account the growth and development of individual students so that the teacher can determine the suitability of the program and make adjustments to it when necessary.

   - The rich data that result from using a variety of assessment methods over time should allow the teacher to make judgments about the progress of the student and communicate this information to others.

6. Evaluation approaches should be manageable for the teacher and provide information for the students and the parents as well as the teacher.

   - Evaluation should be seen as a part of normal teaching activity and not as an additional burden.

   - Each class period is an opportunity for some form of formal evaluation that results in a recorded assessment. However, this does not mean that each student needs to be evaluated on every activity, every period. A workable rotation scheme should be used.

   - It is desirable to use evaluation techniques (such as observation checklists or interviews) that are quickly and easily applied while the students are involved in activities.

   - Accurate and detailed records, including anecdotal comments, should be kept in such a way that they are meaningful to students and parents.

   - Students should be encouraged to think about their own progress. Evaluation activities should provide opportunities for students to learn more about mathematics and about themselves.

STANDARDS

Standards in education are statements that clearly and concisely communicate the level of achievement required to attain a specific learner expectation. Standards determine the extent to which learner expectations must be achieved to be considered successful or the extent to which students must know the content and have the skills required to pass the course. Learner expectations

outline what students are expected to learn. Standard statements establish how well it should be learned.

Teachers often do not consciously establish standard statements, but each teacher must determine how difficult to make a test or assignment, how much homework they expect a student to undertake, and what is considered satisfactory student work both within the class and on assignments. To set their standards, teachers rely on their teaching experience, their training, the experience of other teachers, the level of expectations expressed in basic learning resources and the reactions of students.

In fairness to students, each student across the province who receives 50% in a Grade 11 mathematics course should have to meet the same standard of achievement as any other student with 50% in the course. Currently, consistency between classrooms and schools across the province occurs as a result of the commonality of teacher experience. It is Alberta Education's intention to help teachers establish standards in mathematics. Last fall, Student Evaluation Branch sent out sample standard statements for Mathematics 30 in the Mathematics and Sciences Bulletin 1989-90 School Year. It is expected that the courses of study for Mathematics 30 and 33 will contain standard statements.

EVALUATION STRATEGIES

Interview A planned interview with a student or a group of students is an effective technique for assessing knowledge, understanding, thinking style or attitude; for assessing communication skills (e.g., verbal vs. non-verbal ability); and for learning about personal interests. A well-conducted interview will also give students an opportunity to reflect on their own learning.

Interviews should have a definite purpose and both the teacher and the students should be aware of that purpose. The interviews must be planned in advance. Teachers should be prepared to ask additional leading or key questions to guide discussion, to probe for understanding and/or to correct misconceptions.

A one-on-one interview, conducted with a single student, may be used to assess individual students while small group interviews may be used for assessing instructional effectiveness and for consolidating concepts and skills.

Observation Teachers observe all the time. When observations are documented their effectiveness as an evaluation strategy increases immensely. Documented observations often provide the raw data required for analysis and diagnosis, and provide the basis on which to make remediation or enrichment decisions.

Unlike the interview, observation is a passive strategy. The teacher observes students at work, looking for specific behaviours or outcomes. Documentation may occur in the form of anecdotal records or checklists. Some elements that may be monitored are understanding of concepts or generalizations (e.g., when using manipulatives or in group discussions), enthusiasm, willingness to participate or share ideas, perseverance and independence.

Checklist A checklist is a documentation strategy and is used in conjunction with other evaluation strategies. Checklists can easily be created and customized to meet many different needs and situations. Generally, a matrix is created, listing indicators of desirable behaviours or outcomes on one side, and listing ratings, skill levels or evaluative comments along another side. As teachers note a particular behaviour, they need only check the appropriate column that evaluates or rates that behaviour.

Checklists lend themselves very well to documenting such elements of the program as understanding of concepts using manipulatives; mastery of knowledge, skills or objectives; work habits; organizational skills; problem-solving strategies; and cooperative skills.

Anecdotal Records

Anecdotal records refer to the spontaneous documentation of notable behaviour, effort, achievement, attitudes, changes in performance, social skills or communication skills. Records may be kept in such locations as a daily or weekly diary, individual student files, a specific location in the marks record book, or in a common file that serves as a collection of short dated notes.

Anecdotal records provide specific and dated information that can form the basis for conclusions and assessments. These records often prove invaluable in clarifying assessments and add credibility to observations and recommendations being offered in student, parent and/or teacher meetings.

Written Assessments

Traditional multiple choice and open-ended paper-and-pencil assessments fall into this category.

Two other types of written assessments should be used. One is project writing, where students may report on a particular interest or research. The evaluative technique and grading system used in this case is not unlike the one used in language arts or social studies.

The second is diagnostic writing. Writing assignments that respond to specific mathematics questions, in an expressive writing style, have proven successful as a diagnostic tool in mathematics. Written responses often force students to examine their own understanding of concepts and will communicate to teachers how much students really know about a concept. Written responses also provide insight to how much understanding students have and how they think. For example, students may be asked to give a non-mathematical example of the inverse rule ($a - +b = a + -b$). The response will give teachers different information about student understanding than will 10 mathematics questions like $3 - 4 =$ __. Written responses can be kept in a student diary or logbook, and may be assigned on a regular basis (once or twice a week), or in lieu of a regular quiz.

Peer Evaluation and Self-Evaluation

Involving students in evaluation involves them in their learning. In self-evaluation, the student reflects on their own learning and can come to establish goals for their own improvement. Skills in self-evaluation are important when students leave the classroom environment. Conferences can include self-evaluation. Teacher questions can be effective for initiating self-evaluation (How did you go about solving this problem? What are the strengths of your solutions?)

Peer evaluation involves students in finding value in the mathematical task and developing a sense of problem solving as collaborative work. Checklists are useful to focus peer evaluation and self-evaluation.

For samples of student evaluation ideas, see Appendix C.

WRITING IN MATHEMATICS

Communication, both written and oral, plays a vital role in the implementation of this mathematics program. Students should not only be involved in group problem-solving activities where they must share ideas with one another but should also have the opportunity to express their ideas in writing.

Writing in mathematics requires students to reflect on their reasoning, enhancing cognitive development. Writing also helps the teacher: more than just filling in answers, it reveals the student's level of understanding. Since you will have a good idea of where your students "are" (in their understanding), you will have the background information necessary to plan lessons that promote mathematical thinking and understanding.

This type of writing – writing to describe thinking and reasoning – will be new to students. Hence, students will need to spend time in going "beyond the right answer." This emphasis on writing will help students obtain the goal of mathematical literacy as it promotes thinking and understanding about how a problem is solved.

For more information and examples of writing in high school mathematics, see Appendix D.

COURSE SEQUENCES

To address the diverse needs, abilities and interests of students, the Senior High Mathematics Program is composed of four sequences. Each of the sequences has the development of mathematical literacy as a primary goal and presents curricular content within a problem-solving context.

MATHEMATICS 16/26

The Mathematics 16/26 sequence is designed for students in the Integrated Occupational Program. Upon successful completion of Mathematics 16, students will have fulfilled the mathematics requirement for qualification for the Certificate of Achievement.

MATHEMATICS 14/24

The Mathematics 14/24 sequence is designed for students whose needs, interests and abilities focus on basic mathematical understanding. The emphasis is on the acquisition of practical life skills and students are provided with opportunities to improve their skills in working with mathematics. Students who successfully complete Mathematics 24 may choose to enter directly into a job or select from a limited number of trade programs. The mathematics requirement for the General High School Diploma consists of two courses in mathematics; e.g., Mathematics 14 and Mathematics 24.

MATHEMATICS 13/23/33

The Mathematics 13/23/33 sequence covers material from the areas of algebra, geometry, trigonometry, statistics and consumer mathematics. It is designed for students who require mathematics to prepare them for community and technical colleges, trades and employment. Upon successful completion of Mathematics 23, students will have fulfilled the mathematics requirement for qualification for the General High School Diploma.

MATHEMATICS 10/20/30

The Mathematics 10/20/30 sequence is designed for students with an interest and aptitude in mathematics, who are intending to pursue post-secondary studies at a university or in a mathematics-intensive program at a technical school or college. Mathematics 10/20/30 emphasize the theoretical development of topics from the areas of algebra, geometry, trigonometry and statistics up to a level acceptable for entry into universities and other post-secondary institutions. Upon successful completion of Mathematics 30, students will have fulfilled the mathematics requirement for the Advanced High School Diploma.

MATHEMATICS 31

This course is designed for highly motivated, academic students who wish to pursue a first course in calculus and vectors. Mathematics 30 must be taken as a pre- or a co-requisite course.

COURSE SEQUENCE TRANSFER

According to their interests, needs and abilities, students who have been promoted to high school may choose to begin their high school mathematics education by enrolling in any of the Grade 10 mathematics courses. Some students may wish to complete a basic Grade 10 mathematics course to increase the likelihood of success in a more challenging course. The provision of course transfer enables students to change course sequences, when necessary, to more appropriate levels of challenge.

A student who has achieved a mark of 50% or higher in a given course shall be eligible to take the next or higher ranking course in that sequence.

The local school authority shall have a policy that states clearly the criteria to be met by a student who wishes to change program routes.

When counselling a student to use the course transfer provisions the following circumstances should be considered:

- the student's ability and motivation
- the best interest of the individual student
- judgments are made on an individual basis, not for an entire class of students.

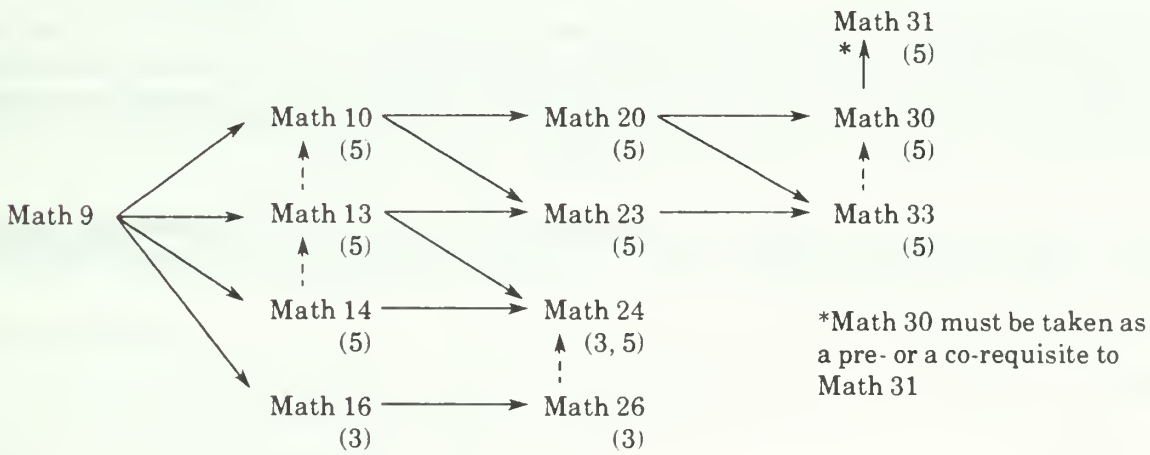
When transferring from a 10/20/30 sequence to a 13/23/33 sequence, or from a 13/23/33 sequence to a 14/24 sequence, the student should transfer into the sequence at the next grade level (for example, from Mathematics 20 to Mathematics 33, or from Mathematics 13 to Mathematics 24). When a student transfers into a less academic sequence, the principal may waive the normal prerequisite for students with less than a 50% standing.

Upon successful completion of Mathematics 33, students may wish to enrol in Mathematics 30 to qualify for the Advanced High School Diploma. Students seeking to make this type of transfer should be aware of the increase in academic demand in Mathematics 30 and should be able to demonstrate a level of achievement in Mathematics 33 that would make success in Mathematics 30 probable. Although less common, some students may wish to enrol in Mathematics 20 after completing Mathematics 23 or in Mathematics 13 after Mathematics 14 or in Mathematics 24 after Mathematics 26. The school principal must waive the normal prerequisite for students wishing to make this type of transfer. Students with credits in a course at lower grade level are not eligible for credits for waived prerequisite courses. For example, students who complete Mathematics 13/23/33 and successfully complete Mathematics 30 are not eligible for credits for the waived prerequisite courses Mathematics 10 and 20 unless they actually take the course.

Please consult the *Guide to Education: Senior High School Handbook 1990–91* (p. 52–54) for further information on course sequence transfer and awarding of retroactive credits.

Provincially developed programs have been designed to accommodate transfer between course sequences at particular points. In the following chart, normal prerequisites are indicated by a solid line and recommended transfer points are indicated by a broken line. Special circumstances may warrant student transfer at other points in the curriculum. All decisions about student transfers between course sequences are made at the local school level since this is where the best interest of the individual student can be determined.

Senior High Mathematics – Course Sequence



*Math 30 must be taken as a pre- or a co-requisite to Math 31

(Credit Values Indicated in Brackets)

LEARNING RESOURCES

INTRODUCTION

Learning resources fall into three categories: Basic, Support and Other learning resources. These categories include print, non-print and electronic software materials used by teachers or students to facilitate teaching and learning.

Basic Learning Resources

Basic learning resources are those student learning resources authorized by Alberta Education as the most appropriate for addressing the majority of learning expectations of course(s), substantial components of course(s), or the most appropriate for meeting general learner expectations across two or more grade levels, subject areas or programs as outlined in provincial programs of study. These may include any resource format; e.g., print, computer software, manipulatives or video.

Support Learning Resources

Support learning resources are those student learning resources authorized by Alberta Education to help address some of the learner expectations of course(s) or components of course(s); or help meet the learner expectations across two or more grade levels, subject areas or programs as outlined in the provincial programs of study.

Other Learning Resources

Other learning resources are those learning resources identified by Alberta Education as useful for teachers in the implementation of a course(s) or program(s) of studies, but which have not undergone the standard review procedures of Alberta Education. Alberta Education does not accept responsibility for use of these resources with students. A disclaimer statement is used in teacher support publications whenever these resources are listed.

The following learning resources support the Mathematics 30/33 courses. The abbreviations for the resources, used throughout the TRM, are noted in brackets. A description of the Support and Other resources can be found in Appendix B.

MATHEMATICS 30

Proposed Basic

Bye, M., Holtmath 12. HBJ-Holt Publishing, 1988.

Ebos, F. et al., Mathematics: Principles and Process 11. Nelson, 1991.

Kelly, B. et al., Mathematics 12. Addison Wesley, 1988.

Landwehr, J. et al., Exploring Surveys and Information from Samples. Dale Seymour Publications, 1985.

Landwehr, Watkins, Exploring Data. Dale Seymour Publications, 1986.

Support

Algebra Tiles – Student Set.

American Statistical Association, Quantitative Literacy Software.

Charles, R. et al., How to Evaluate Progress in Problem Solving. Reston, Virginia: The National Council of Teachers of Mathematics, 1987.

Demana, F. et al., Master Grapher. Don Mills, Ontario: Addison-Wesley Publishers Ltd., 1988.

Hirsch, C. R. (ed), Activities for Implementing Curricular Themes from the Agenda for Action. Reston, Virginia: The National Council of Teachers of Mathematics, Inc., 1986.

Howden, H., Algebra Tiles for the Overhead Projector. New Rochelle, New York: Cuisenaire Company of America, 1985.

James, G. et al., Mathematics Dictionary Fourth Edition. New York, New York: Van Nostrand Reinhold Company, 1976.

National Council of Teachers of Mathematics, Curriculum and Evaluation Standards for School Mathematics. Reston, Virginia: National Council of Teachers of Mathematics, 1989.

Swan, M., The Language of Graphs. Nottingham, England: The Shell Centre for Mathematical Education, 1987.

Other: see Appendix B.

MATHEMATICS 33

Proposed Basic

Carli, Mathematics for a Modern World Book 4. Gage, 1985.

Ebos, F. et al., Math Matters Book 4. Nelson, 1991.

Landwehr, J. et al., Exploring Surveys and Information from Samples. Dale Seymour Publications, 1985.

Landwehr, Watkins, Exploring Data. Dale Seymour Publications, 1986.

Support

Algebra Tiles: Student Set.

American Statistical Association, Quantitative Literacy Software.

Charles R., et al., How to Evaluate Progress in Problem Solving. Reston, Virginia: The National Council of Teachers of Mathematics, 1987.

Demana, F. et al., Master Grapher. Don Mills, Ontario: Addison-Wesley Publishers Ltd., 1988.

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National Council of Teachers of Mathematics, Curriculum and Evaluation Standards for School Mathematics. Reston, Virginia: National Council of Teachers of Mathematics, 1989.

Swan, M., The Language of Graphs. Nottingham, England: The Shell Centre for Mathematical Education, 1987.

Other: see Appendix B.

HOW TO USE THE TEACHER RESOURCE MANUAL

This Interim Teacher Resource Manual is designed to assist teachers in implementing the Mathematics 30/33 courses. It is specifically designed to be used on a regular basis. The design of the manual is as follows:

CONCEPT 1

The concept box includes the specific learner expectations taken from the course of studies.

COMMENTS

The **comments** section includes: comments to help clarify the intent of specific learner expectations; an overview of the material that students may have covered in previous courses; notes regarding material that may be considered extraneous; and limiting examples.

PROCESS/PROBLEM-SOLVING CONTEXT

Suggestions for activities and examples of problems that are appropriate to the specific learner expectations are included, as well as references to other books and articles applicable to the specific learner expectations. Periodically, there will be problems or activities that apply to specific learner expectations; they will be preceded by **"Activity/Problem 2.2"** (this activity or problem refers to learner expectation 2.2). If the comment is preceded by **"Activity/Problem"** then the comment refers to the whole concept.

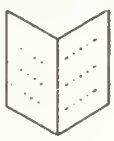
TECHNOLOGY INTEGRATION

Suggestions for specific calculator and computer exercises, video programs and for authorized computer software are included.

ELECTIVE SUGGESTIONS

In some cases, elective suggestions are provided. This includes problems and activities that are extensions of the specific learner expectations.

RESOURCE CORRELATION



Provides a correlation of the Basic and Support textbooks that are approved for the Mathematics 30/33 courses.

MATHEMATICS 30

PROGRAM EMPHASIS

Required Content

The major part of the content of each high school mathematics course consists of topics required of all students who take the course. The **required content** comprises 80% of the course and contains the concepts, skills and attitudes that all students are expected to acquire. As well, the required portion of all courses includes specific expectations for problem solving and the use of technology.

Elective Material

Each course includes a compulsory component comprising 20% of the course, made up of **elective material** that is consistent with the content and expectations of their required component. The purpose of the elective material is to provide for enrichment, remediation, or innovative experimental presentations or activities. It is not intended to provide acceleration or advanced placement. However, horizontal enrichment and extension is appropriate and students should have access to elective material that serves their individual needs and interests.

Suggested Program Emphasis (in %) for Required Portion of Mathematics 30

The following are suggestions only to guide teachers with their planning. Individual students may require more or less effort to successfully complete a particular unit of study.

Unit	Suggested Program Emphasis (in %)
Polynomial Functions	10
Trigonometric and Circular Functions	15
Statistics	15
Quadratic Relations	10
Exponential and Logarithmic Functions	10
Permutations and Combinations, Binomial Theorem	10
Sequences and Series	10
Elective	20

ATTITUDES

Students will be expected to demonstrate an attitude associated with mathematical literacy. In particular, *students will be expected to*:

- be confident in their mathematical knowledge and in their ability to acquire new knowledge;
- demonstrate persistence, resolve, flexibility and ingenuity in finding the solution to problems;
- develop intellectual curiosity and openness to new ideas, insights and change in the pursuit of mathematical knowledge;
- exhibit an attitude of curiosity and spontaneity and appreciate creativity and innovation in representing situations mathematically;
- be critical and constructive in approaching new ideas and new processes;
- be aware of the importance of communication skills in mathematics;
- appreciate the usefulness of computational competence, mathematical processes and problem-solving skills that are used in the decision-making and modelling processes in our society;
- appreciate the contributions of mathematics to our culture and civilization.

PROBLEM SOLVING

Students will be expected to use a variety of procedures to help them understand mathematical problems. In particular, they will be expected to:

- read the problem thoroughly
- identify and clarify key components
- restate the problem using familiar terms
- evaluate the given information as to whether it is insufficient or extraneous
- interpret pictures, charts and graphs
- determine any hidden assumptions
- ask relevant questions
- identify given, needed and wanted information
- diagram or model the problem situation
- use suitable notation
- determine valid inferences.

Students will be expected to develop a variety of strategies for use in the solution of mathematical problems. In particular, they will be expected to:

- conduct an investigation
- use estimation and approximation
- develop equations or use formulas
- use flow charts
- make lists and charts
- look for patterns
- work backward
- break the problem into smaller parts
- look for a simpler or related problem
- make diagrams or models
- use manipulatives
- choose and sequence a series of mathematical operations

- sketch the graph of a problem situation
- establish procedures to gather and organize data.

Students will be expected to develop a variety of skills that can be used to carry out the plan for the solution of a problem. In particular, they will be expected to:

- apply selected strategies.
- present ideas clearly
- document the solution process
- use appropriate group behaviours
- use calculators and computers
- evaluate problem-solving strategies for effectiveness
- search for additional information
- ask questions
- be open to inspirations, intuitions and "bright ideas."

Students will be expected to employ a variety of skills to help them look back over the solution of a problem. In particular, they will be expected to:

- determine the reasonableness of an answer
- explain the solution in oral or written form
- consider the possibility of additional solutions
- search for other strategies and processes of solution
- create and solve similar problems
- note the characteristics that will be identifiable in similar problems
- make a generalization.

POLYNOMIAL FUNCTIONS
SUGGESTED PROGRAM EMPHASIS – 10 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that a polynomial function is a function of the form $f(x) = a_x x^n r a_{x-1} x - 1 \ldots ax + a_0$, where $a_0, a_1, \ldots a_x$ are real numbers and $x \in N$.		
Students will be expected to demonstrate an understanding that a polynomial function can be graphed on a Cartesian plane and that such graphs will have particular characteristics dependent on the function.	Students will be expected to sketch the graphs of integral polynomial functions.	Students will be expected to draw the graphs of integral polynomial functions using calculators or computers. Students will be expected to investigate the characteristics of the graphs of polynomial functions of different degrees and determine the effects of a multiplicity of zeros on the graphs of polynomial functions. Students will be expected to find approximations for the zeros of integral polynomial functions using calculators or computers. Students will be expected to analyse points on the graphs of polynomial functions using calculators or computers. Students will be expected to solve problems that can be represented by polynomial functions.
Students will be expected to demonstrate an understanding that many polynomial functions can have the same zeros.	Students will be expected to derive an equation of an integral polynomial function given its zeros. Students will be expected to derive the equation of an integral polynomial function given its zeros and an ordered pair that satisfies it.	Students will be expected to find the equation of a polynomial function given its zeros and any other information that will uniquely define it.
Students will be expected to demonstrate an understanding of the following form of the division algorithm for polynomials: If any polynomial $P(x)$ is divided by a binomial of the form $(a - x)$ (called $D(x)$), the result will be a polynomial quotient $Q(x)$ and a remainder R.	Students will be expected to divide integral polynomial functions in one variable by a binomial of the form $(x - a)$, $a \in Q$. Students will be expected to write the division operation on a polynomial function by a binomial in the form of the Division Algorithm: $P(x) = D(x)\,Q(x) + R$.	

POLYNOMIAL FUNCTIONS
SUGGESTED PROGRAM EMPHASIS – 10 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that when a polynomial P(x) is divided by a binomial of the form $(x-a)$, the remainder, R, is equal to P(a) (Remainder Theorem).	Students will be expected to use the Remainder Theorem to evaluate polynomial functions for rational values of the variable. Students will be expected to use the Factor Theorem to factor an integral polynomial function completely and to determine all of its real zeros.	Students will be expected to prove the Remainder Theorem. Students will be expected to use the Remainder Theorem to prove that if a number "a" is a zero of a polynomial function P(x) then $(x-a)$ will be a factor of P(x) (Factor Theorem). Students will be expected to use technology to factor polynomial functions. Students will be expected to recognize that all rational zeros of a polynomial function will be the form p/q, where p is a factor of a_0 and q is a factor of a_n.

CONCEPT 1

Students will be expected to demonstrate an understanding that a polynomial function is a function of the form $f(x) = a_x x^n r a_{x-1} x - 1 \ldots ax + a_0$, where $a_0, a_1, \ldots a_x$ are real numbers and $x \in N$.

COMMENTS

This concept can be related back to previous knowledge, for example, quadratics in Mathematics 20.

A review of terminology associated with polynomial functions may be necessary. However, review these terms as they appear. A list of vocabulary to memorize at the beginning of a unit serves little purpose.

It should be mentioned that polynomial functions may be expressed in variables other than "x" and "y" and may represent physical phenomena.

Students could be asked to write examples of polynomial functions as well as examples of non-polynomial functions.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Have students write formulas from physics or previous math courses that are polynomial functions. For example, the surface area of a cylinder with a height of 10 cm, the area of a sphere, the distance of a particle from a fixed point ($f(t) = 40 - 15t + 4t^2 + 3t^3$. Note that this is a specific situation.) These formulas could then be evaluated for different values of the variable.

TECHNOLOGY INTEGRATION

Use a calculator or spreadsheet to evaluate expressions.

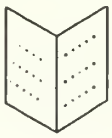
EXTENSION

Given a table of values, write the function that represents the data. Use techniques such as trial and error and finite differences to find functions. (For the method of using finite differences see Mathematics Teacher, Volume 79, Number 6, September 1986.) Use the formulas from the sequences and series unit if you have taught this unit before polynomials.

If this unit has been introduced using the graphs of polynomial functions, then a table of values could be determined from the graph and checked by substitution into the function.

By analyzing the graph of the function a student could predict what the maximum and/or minimum value of a polynomial could be, what value of "x" is associated with this value and the values for "x" that will give the polynomial a value of positive, negative or zero.

RESOURCE CORRELATION



HM12:	pp. 90–91
MPP12:	pp. 142–144
M12:	pp. 42–48

CONCEPT 2

Students will be expected to demonstrate an understanding that a polynomial function can be graphed on a Cartesian plane and that such graphs will have particular characteristics dependent on the function.

Students will be expected to:

2.1 sketch the graphs of integral polynomial functions
- 2.1.1 draw the graphs of integral polynomial functions using calculators or computers
- 2.1.2 investigate the characteristics of the graphs of polynomial functions of different degrees and determine the effects of a multiplicity of zeros on the graphs of polynomial functions
- 2.1.3 find approximations for the zeros of integral polynomial functions using calculators or computers
- 2.1.4 analyse points on the graphs of polynomial functions using calculators or computers
- 2.1.5 solve problems that can be represented by polynomial functions.

COMMENTS

NCTM's Curriculum and Evaluation Standards for School Mathematics (1989), states that "calculators and computers with appropriate software transform the mathematics classroom into a laboratory . . . where students use technology to investigate" and "the teacher encourages experimentation" (p. 128). The properties of polynomials and their graphs can be effectively dealt with by presenting general problems that students attempt to answer by using graphing software.

The polynomial functions of this unit are limited to those with integral coefficients (i.e., no rationals or irrationals as coefficients).

Once the students are familiar with the relationship between zeros and x-intercepts they should be introduced to the idea of zeros with multiplicities and the influence they have on the graph.

e.g., $y=(2x-1)(x-2)^2(x+7)^3$ has six zeros:
-7 with a multiplicity of 3
$+2$ with a multiplicity of 2
$+1/2$ with a multiplicity of 1

It is sometimes difficult to see the effect of multiplicities of three without the use of graphing software.

Students should be able to recognize the degree of a polynomial from its graph. They should also be aware of the effect of the sign of the highest powered coefficient on the shape of the growth. Again the students should be allowed to discover these effects and be encouraged to write down their findings or at least share them with the group.

Students can be encouraged to explore on their own by asking "what-if" questions like:

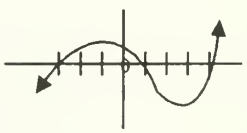
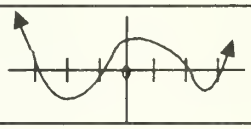
What if the highest power coefficient is negative?
What if the coefficients are sequential?
(e.g., $y = 1x^3 + 2x^2 + 3x + 4$)

Students should understand the connection between polynomial equations and functions. By graphing the corresponding function and looking for the x-intercepts it is possible to approximate the roots of any polynomial equation. This is particularly useful if the polynomial does not factor.

Students do not require a knowledge of the representation of complex numbers. (i.e., the "i" notation is beyond this course.) The existence of complex roots should be explained by the presence of factors like $(x^2 + 1)$ that have no real solutions. The effect of complex roots on the graph should be understood.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: The student is asked to fill out a table like the one below using a graphing calculator or graphing program. (This activity could reinforce or introduce the relationship between factors and x-intercepts.) The students work in groups and are asked to write about the relationship between the columns.

Function in Factored Form	Sketch	Function in Expanded Form
$y = (x-1)(x+2)(x-3)$		
$y = x(x+1)(x-4)(x+5)$		
		$y = x^3 + 3x^2 - 10x - 24$
		$y = x^3 + 11x^2 + 23x + 35$
		
		

Activity: Again the students are asked to use the technology available to fill all the columns below. (The first one is done for them.) Before they begin they may need to be reminded of what they know about the relationship between a quadratic function and its corresponding equation. (From Mathematics 20.)

Function	Corresponding Equation	Solution(s) of Equation
$y=x^3-9x+x^2-9$	$x3-9x+x^2-9=0$	$x=-3, -1, +3$
$y=x^4-9x^2+3x+12$		
$y=x^4+6x^3+7x^2-6x-8$		
		$x=-1, 5, 8$
		$x=0, -3, 1$
	$(x-1)(x+2)(x-5)=0$	

Problem: Write any third degree polynomial where the coefficient of the x^3 term is equal to one, that is not factorable. Check by graphing.

Problem: Solve the polynomial equation corresponding to the function in the problem above (to three decimal places). This may require some special knowledge of the calculator being used. This is also a good time to use some of the BASIC language programs given in many textbooks for just such a purpose. (Mathematics 12, pp. 68–70; Holtmath 12, p. 112; Mathematics Principles and Process, p. 167.) A spreadsheet can also be used to answer this question.

Activity: Predict the shape of the graph of polynomial functions of degree 5 and 6. Verify using specific examples.

Activity: Create various polynomials with multiplicities of 2 and 3. Graph and write a description of the effect. (The teacher may wish to mention the term inflection point.) Students will naturally ask about the effect of higher multiplicities. They should be encouraged to explore on their own.

Problem: Write an integral polynomial function that has two complex and one rational zero of multiplicity two. (Verify by graphing.)

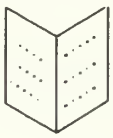
Problem: Write an integral polynomial that has two irrational zeros, two complex zeros and passes through the point (2, 10).

Activity: Produce a flow chart that shows the steps that someone could use to sketch a graph of a polynomial function (from the initial steps of factoring the polynomial) assuming that they do not have a calculator that graphs.

Activity: Solve Mathematics 31 calculus maximum/minimum problems without the derivative by graphing the function involved and approximating where the maximum or minimum occurs. Choose questions where the function is given and also where the student must derive a function.

As an example the question on p. 53 of the Mathematics 31 text gives the formula $V = 4x^3 - 44x^2 + 96x$. The student could be asked what values give a volume of zero and which give a maximum, and to find the maximum value. (Also refer to any introductory calculus text on the topic of maximums and minimums.)

RESOURCE CORRELATION



M12:	pp. 42–45; 63–76
HM12:	pp. 104–107
MPP12:	pp. 151–155

CONCEPT 3

Students will be expected to demonstrate an understanding that many polynomial functions can have the same zeros.

Students will be expected to:

3.1 derive an equation of an integral polynomial function given its zeros

3.2 derive the equation of an integral polynomial function given its zeros and an ordered pair that satisfies it

3.2.1 find the equation of a polynomial function given its zeros and any other information that will uniquely define it.

COMMENTS

Students may not be aware that the zeros of a polynomial do not uniquely determine its equation. For example, $y=(x-1)(x+3)(x+2)$ and $y=3(x-1)(x+3)(x+2)$ both have the same zeros. Students can be introduced to these ideas by comparing first and second degree functions that are related in this fashion (e.g., compare $y=x+2$ and $y=2x+4$ or $y=x^2+3x$ and $y=2x^2+6x$.) By allowing the students to graph with the technology available they will be able to draw correct conclusions on their own.

In this section they should be reminded of the work done in Mathematics 33 and 20 regarding the transformation of functions (e.g., the relationship between f(x) and cf(x) where c is a Real number).

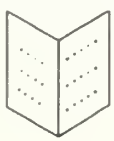
PROCESS/PROBLEM-SOLVING CONTEXT

Problem: Solve the system given by $y=(x-1)(x+3)(x-2)$ and $y=2(x-1)(x+3)(x-2)$. In this problem they should see that the graphs, although similar in many ways, have only three points in common.

Activity: Compare the graphs of the functions given by cf(x) where $f(x)=x^3-3x+2$ and c takes the values 1, 2, and -3. Write a brief paragraph describing their similarities and differences.

Problem: Design a function that has certain predetermined characteristics and then verify the answer by graphing. An example would be to find a polynomial function with zeros of 1, 2 and -3 that goes through the point (5, 192).

RESOURCE CORRELATION



HM12:	pp.
MPP12:	pp. 156
M12:	pp. 74–78

CONCEPT 4

Students will be expected to demonstrate an understanding of the following form of the division algorithm for polynomials: If any polynomial P(x) is divided by a binomial of the form (a − x) (called D(x)), the result will be a polynomial quotient Q(x) and a remainder R.

Students will be expected to:

4.1 divide integral polynomial functions in one variable by a binomial of the form $(x-a)$, $a \in Q$

4.2 write the division operation on a polynomial function by a binomial in the form of the Division Algorithm: $P(x) = D(x)\,Q(x) + R$.

COMMENTS

Students have previously learned long division by binomials in Mathematics 10. Students from Mathematics 33 may not have encountered division by a binomial so will require some assistance in this area. Do not spend a lot of time teaching this skill.

Students should not be expected to spend too much time on any one method of dividing polynomials. It is important that students understand that any polynomial can be considered the product of two polynomials called the dividend and the quotient plus another polynomial called the remainder. This concept builds into the Remainder Theorem. In keeping with the emphasis in the revised program the actual use of an algorithm such as synthetic division need not be stressed. Students should be encouraged to use a variety of methods to find the factors of a polynomial such as graphing the functions using a graphing calculator or computer package, or use of calculators or spreadsheets to evaluate polynomials at a number of different values. These methods are much more powerful techniques than those that use division algorithms since these methods can easily handle simple as well as complex functions.

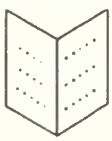
PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Write 458 as a product of two whole numbers plus another whole number. Explain how you arrived at your answer.

Write $3x^3+2x^2-x+5$ in five different ways in the form $D(x)Q(x)+R$ where D(x) is of the form $x-a$ and a is an integer.

Write x^3-x-2 in five different ways in the form $D(x)Q(x)+R$ where D(x) is of the form $x-a$ and a is an integer. One of the ways should have $R=0$.

RESOURCE CORRELATION



HM12: pp. 92–93
MPP12: pp. 149–150
M12: pp. 49–53

CONCEPT 5

Students will be expected to demonstrate an understanding that when a polynomial P(x) is divided by a binomial of the form (x – a), the remainder, R, is equal to P(a) (Remainder Theorem).

Students will be expected to:

5.1 use the Remainder Theorem to evaluate polynomial functions for rational values of the variable
- 5.1.1 prove the Remainder Theorem
- 5.1.2 use the Remainder Theorem to prove that if a number "a" is a zero of a polynomial function P(x) then (x – a) will be a factor of P(x) (Factor Theorem)

5.2 use the Factor Theorem to factor an integral polynomial function completely and to determine all of its real zeros
- 5.2.1 use technology to factor polynomial functions
- 35.2.2 recognize that all rational zeros of a polynomial function will be the form p/q, where p is a factor of a_0 and q is a factor of a_n.

COMMENTS

As mentioned in other concepts, the interpretation of graphs should lead the students to some interesting insights into the relationships among graphs, zeros and factors. Calculators and computers can be instrumental in providing the graphical information to allow students to make these generalizations. However, any scientist and mathematician will readily maintain that there is a great difference between a generalization and a formal proof. Part of this concept involves helping students realize how to formalize relationships that seem apparent.

PROCESS/PROBLEM-SOLVING CONTEXT

The Remainder Theorem provides an excellent opportunity for group work and for the students to write their own conclusions. An intermediate step before the actual theorem is developed and proved would be a level of abstraction wherein the student applies the conclusions that they have written for a problem like the following:

compare the remainder of $(x^2+2x-1)\div(x-b)$ with P(b) if $P(x)=x^2+2x-1$.

The proof for the Remainder Theorem is not provided in some of the texts so it is given below:

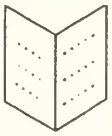
From the division algorithm, $P(x)=D(x)\bullet Q(x)+R$
If P(x) is divided by x – a, then $P(x)=(x-a)\,Q(x)+R$
If we set x = a, then $P(a)=(a-a)\,Q(x)+R$
$P(a)=0Q(x)+R$
Therefore, $P(a)=R$.

The students, having progressed through numerical answers to literal answers, should be involved in this development.

TECHNOLOGY INTEGRATION

As has been mentioned previously, graphing calculators and computers have greatly changed the way in which this unit can be studied; their use is highly recommended. Time should be taken to illustrate how polynomials can be evaluated using a variety of methods including the following: spreadsheets with computers, programmable calculators and using the memory functions of more basic calculators.

RESOURCE CORRELATION



HM12:	pp. 94–95; 98–103
MPP12:	pp. 145–153
M12:	pp. 54–63

TRIGONOMETRIC AND CIRCULAR FUNCTIONS
SUGGESTED PROGRAM EMPHASIS – 15%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that the radian measure of an angle is the ratio of the arc it subtends to the radius of a circle in which it is a central angle, and that one radian is the measure of a central angle subtended in a circle by an arc whose length is equal to the radius of the circle.	Students will be expected to measure central angles in a circle using radian measure. Students will be expected to convert angle measurements from degree to radian measure and vice versa. Students will be expected to determine the exact values of the trigonometric ratios coterminal with $\frac{n\pi}{6}, \frac{n\pi}{4}, \frac{n\pi}{3}, \frac{n\pi}{2}$, and $n\pi$, $n \in I$.	
Students will be expected to demonstrate an understanding that identities are statements of equality that are true for all values of the variable and that trigonometric identities are equations that express relations among trigonometric functions that are valid for all values of the variables for which the functions are defined.	Students will be expected to use the trigonometric identities: Reciprocal Identities $\frac{1}{\sin\theta} = \csc\theta$ $\frac{1}{\cos\theta} = \sec\theta$ $\frac{1}{\tan\theta} = \cot\theta$ Quotient Identities $\frac{\sin\theta}{\cos\theta} = \tan\theta$ $\frac{\cos\theta}{\sin\theta} = \tan\theta$ Pythagorean Identities $\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$ Students will be expected to use the addition and subtraction identities (formulas): $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin a \sin\phi$ $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos a \sin\phi$	Students will be expected to derive the quotient and Pythagorean identities using deductive processes. Students will be expected to use the fundamental trigonometric identities to simplify, evaluate and prove trigonometric expressions involving identities.
Students will be expected to demonstrate an understanding that trigonometric functions can be graphed on a Cartesian plane.	Students will be expected to graph the following forms of the sine, cosine and tangent functions: $y = a \sin[b(\theta + \phi)] + d$ $y = a \cos[b(\theta + \phi)] + d$ $y = \tan\theta$.	Students will be expected to use electronic calculators or computers to draw and analyse the graphs of trigonometric functions. Students will be expected to investigate the effects of the parameters a, b, c, d on the graphs of the trigonometric functions using calculators or computers.

TRIGONOMETRIC AND CIRCULAR FUNCTIONS
SUGGESTED PROGRAM EMPHASIS – 15%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding of the methods used to solve trigonometric equations.	Students will be expected to solve first and second degree trigonometric equations involving multiples of angles on the domain 0 to 2π. Students will be expected to demonstrate the relationship between the solution of a trigonometric equation and the graph of the corresponding function.	Students will be expected to use electronic calculators or computers to solve trigonometric equations by evaluating the graphs of trigonometric functions.

CONCEPT 1

Students will be expected to demonstrate an understanding that the radian measure of an angle is the ratio of the arc it subtends to the radius of a circle in which it is a central angle, and that one radian is the measure of a central angle subtended in a circle by an arc whose length is equal to the radius of the circle.

Students will be expected to:

1.1 measure central angles in a circle using radian measure

1.2 convert angle measurements from degree to radian measure and vice versa

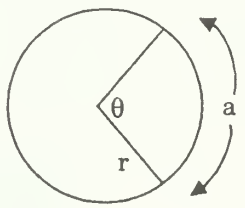
1.3 determine the exact values of the trigonometric ratios coterminal with $\frac{n\pi}{6}$, $\frac{n\pi}{4}$, $\frac{n\pi}{3}$, $\frac{n\pi}{2}$, and $n\pi$, $n \in I$.

COMMENTS

Students have previously worked with degree measures for angles. In Mathematics 20, students learned how to recognize and sketch positive and negative angles in standard position as well as how to determine the trigonometric ratios of any angle by comparing it with the reference angle. A review of this and the special triangles would be useful to introduce concept 1.3.

For situations involving the rotation of objects it is convenient to measure angles in radians.

Definition of radian: A radian is the measure of the central (sector) angle subtended by an arc equal to the radius of the circle. The radian measure of the angle is the ratio of the arc length to the radius ($\theta = a/r$). A central angle has its vertex at the centre of the circle. The arc subtending the angle is the subset of the points of the circle cut off by the arms of the central angle. (See diagram.)



Radian measure is a direct relationship between the arc length and the radius. This enables calculations without continually returning to triangle trigonometry. Applications include questions involving angular velocity, RPMs and distance travelled by a point on a wheel.

The relationship between radian measure and degree measure is an example of direct variation.

Exact values for the trigonometric ratios for angles coterminal with $n\pi/6$, $n\pi/4$, $n\pi/3$, are those resulting from using the sides of a 30°-60°-90° triangle or a 45°-45°-90° triangle and are left in radical form.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have students draw circles on cardboard. Using a piece of string cut to the same length as the radius, wrap the string around the circumference of the circle and draw the central angle that it subtends. Cut out the angle and compare its size to those angles created by other students. Measure the angle. Why is the measure of the angle slightly less than 60°?

Have students draw a circle and using a string cut to the length of the diameter of this circle, draw an angle whose arc length is equal to the diameter. What is the radian measure of the angle?

Have students draw a circle with a 180° angle at the centre. Ask them to devise a method to measure the size of this angle in radians.

Activity 2: Translating between radians and degrees gives students a way of relating radian measure back to something familiar. Begin by asking the question "How big in degrees is 1 rad?" Have students draw a circle with a radius of one (the unit circle) or any size circle if they want. Have them calculate the circumference of this circle. (Keep it exact. $C = 2\pi$ for the unit circle.) Have them find the radian measure for a 360° angle, using the definition of radian measure, and place it in the table of values as shown. Have them calculate in this way the radian measure of the angles to 30°.

Angle in degrees	Angle in radians
360	2π
180	π
90	$\pi/2$
60	
45	$\pi/4$
30	$\pi/6$
1 degree	1 radian

The table represents a direct variation so its defining rule is of the form $D = kR$ where k is the constant of variation. Find the constant of variation and the resulting equation that relates degree measure to radian measure. Use this rule to complete the table.

Activity 3: Have students create a radian protractor. This protractor can be used to compare angles measured in radians and in degrees.

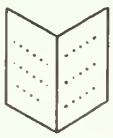
Determining exact values for the trigonometric ratios for angles in radian measure could be determined by using the unit circle and arc lengths associated with the special angles of 30°, 45°, 60°, 90° etc.

Reprinted from National Council of Teachers of Mathematics, Mathematics Teacher, February 1985.

TECHNOLOGY INTEGRATION

Teach students to use the DRG key on a calculator to translate between radian and degree measure.

RESOURCE CORRELATION



HM12:	pp. 287–288; 294–295
MPP12:	pp. 172–190
M12:	pp. 156–170

CONCEPT 2

Students will be expected to demonstrate an understanding that identities are statements of equality that are true for all values of the variable and that trigonometric identities are equations that express relations among trigonometric functions that are valid for all values of the variables for which the functions are defined.

Students will be expected to:

2.1 use the fundamental trigonometric identities:

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\frac{1}{\sin\theta} = \csc\theta$	$\frac{\sin\theta}{\cos\theta} = \tan\theta$	$\sin^2\theta + \cos^2\theta = 1$
$\frac{1}{\cos\theta} = \sec\theta$	$\frac{\cos\theta}{\sin\theta} = \cot\theta$	$\tan^2\theta + 1 = \sec^2\theta$
$\frac{1}{\tan\theta} = \cot\theta$		$\cot^2\theta + 1 = \csc^2\theta$

2.1.1 demonstrate an understanding of the quotient and Pythagorean identities using deductive processes

2.1.2 use the fundamental trigonometric identities to simplify, evaluate and prove trigonometric expressions involving identities.

COMMENTS

Identities are statements that are true for any value of the variable. Remind students that they have already worked with identities when they worked with polynomials (such as in factoring an expression like $x^2-4=(x+2)(x-2)$ or performing basic operations on expressions such as $(3x+4) - (x-2) = 2x+6$.

The reciprocal identities are definitions and can be introduced as such. Mention may be made of their historical significance (when calculation was done by hand, it was easier to multiply decimal numbers from tables than it was to divide by them) but they should not be used to solve triangle problems.

Some students may be interested in the names of some of the functions, particularly tangent and secant and want to know what relationship the trigonometric functions have to the line/circle definitions. These can be explored through the Elective described in this section.

Students should be able to identify identities by substituting arbitrary values for the variable. This is a worthwhile exercise before beginning the proof of most identities. If it can be shown that the expression is not true for a chosen value, then that is sufficient to state that the expression is not true. This provides an excellent opportunity to discuss the meaning of necessary and sufficient conditions for proof.

Manipulating trigonometric identities provides an excellent opportunity for students to learn to "prove" that a given relationship is true. The nature of proof should be discussed, particularly in terms of the difference between a verification using particular values of the variable and a complete argument that demonstrates truth in general. A discussion of the nature of deductive and inductive proofs would fit well here. Students should be shown the difference between working with equations and working with identities, particularly that there are some operations that have been used to solve equations that cannot be used to verify identities. (For example, "cross multiplication" is not a legitimate operation with identities because it assumes that the two sides are equal. In a possible identity, that is not true until it is proved.)

It is felt that proving identities by multiplying complex fractions by conjugal expressions is beyond the scope of this concept.

PROCESS/PROBLEM-SOLVING CONTEXT

Students should be encouraged to present logical arguments to show that the quotient and Pythagorean identities are true. Note that this does not necessitate the use of the T proof processes that were so common in the teaching of deductive geometry. This is an excellent place to discuss deduction and show students that a proof is a logical, cogent sequence of statements beginning with what is given or is known to be true, followed by statements based on previously established knowledge and concluding with what is to be proved. An example of a complete proof of the first Pythagorean identity follows:

Prove that $\sin^2\theta + \cos^2\theta = 1$.

From previous knowledge, $\frac{y}{r} = \sin\theta$ and $\frac{x}{r} = \cos\theta$ for all angles drawn in standard position. Therefore,

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2}.$$

Since we know from previous study that $y^2 + x^2 = r^2$ it follows that $\frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$.

Therefore, $\sin^2\theta + \cos^2\theta = 1$.

Activity: Have students create identities for others to prove or ask them to express a given expression in several equivalent ways. Ask them to express any trigonometric ratio in terms of any other ratio.

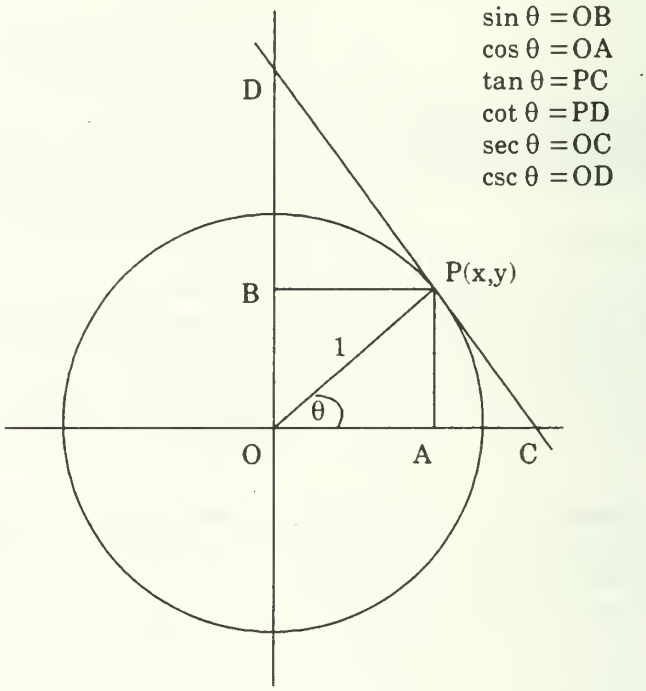
TECHNOLOGY INTEGRATION

As the identities are introduced, students should be encouraged to use their calculators to verify them for various values of the angle, measured in both degrees and radians.

Using graphing calculators or computers, have students graph $\sin^2\theta + \cos^2\theta = y$ and discuss what the graph means. After they have graphed $\sin\theta = y$ and $\cos\theta = y$ in later sections, it will be worthwhile to discuss why this graph is a horizontal line through $y = 1$.

ELECTIVE

An interesting approach to the trigonometric ratios and how they relate to the lines of the same name can be shown through the use of an angle in standard position drawn in a unit circle with a tangent line drawn to the end point of the terminal arm and intersecting the axes. The values of all six ratios for the angle can be determined in terms of the segment lengths on the diagram. For a full explanation see Mathematics Teacher, Volume 78, Number 7, October 1985.



RESOURCE CORRELATION

HM12:	pp. 316–319
MPP12:	pp. 216–220
M12:	pp. 171–173; 230–235

CONCEPT 2 (continued)

Students will be expected to demonstrate an understanding that identities are statements of equality that are true for all values of the variable and that trigonometric identities are equations that express relations among trigonometric functions that are valid for all values of the variables for which the functions are defined.

Students will be expected to:

2.2 use the addition and subtraction identities (formulas):
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$
$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$.

COMMENTS

Students can verify that all the above formulas and identities appear to be true using a calculator and substituting several values for the variables in each expression. Again, the trigonometric function of positive and negative angles (Mathematics 20) can be applied to develop the negative angle identities. Derivation of the sum and difference formulas is not an expectation in this concept, however, such derivations would be appropriate extension activities. Students are expected to prove the negative arc and complementary arc identities. Refer to the previous comment section for a statement on proof. Memorization of the negative arc and complementary arc identities is unnecessary since they can all be derived from the sum and difference formulas.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem: Show that $\sin(\theta+\phi)=\sin\theta+\sin\phi$ is not an identity. Write a logical argument.

In order that the expression be an identity it must be true for all values of a and b. Substitute values for a and b. If one pair of values does not yield a true statement then the expression is not an identity. Students will find such a pair. It is important that students very quickly realize that this expression is not an identity as it probably looks as if it should be one.

Problem: Is $\cos(\theta+\phi)=\cos\theta+\cos\phi$ ever true for any values of θ and ϕ? Give an argument for your answer.

Students could use guess and check. A table of values would help in the organization of data. A spreadsheet could be set up to produce such a table of values very rapidly.

Students may use a calculator or computer to graph such functions as $y=\cos(60°+\phi)$ and $y=\cos 60°+\cos\phi$ to see if they intersect. Other values of "a" could be tried.

Problem: Given that $\sin(\theta-\phi) = \sin\theta\cos\phi-\cos\theta\sin\phi$, show that $\sin(-\theta) = -\sin\theta$.

Proof

$-\theta = 0°-\theta$ so $\sin(-\theta) = \sin(0°-\theta)$

Using the difference identity for sin

$$\begin{aligned}\sin(0°-\theta) &= \sin 0°\cos\theta-\cos 0°\sin\theta\\ &= (0)\cos\theta-1\,(\sin\theta)\\ &= 0-\sin\theta\\ &= -\sin\theta.\end{aligned}$$

TECHNOLOGY INTEGRATION

Calculators can help students reach an intuitive understanding about the correctness of the formulas and identities. A computer spreadsheet also allows students to very quickly produce large tables of values to check the apparent correctness of the expressions. When using a spreadsheet one must take care to change degree measure to radians as the trig functions used in the spreadsheets use radians.

Students could also use a computer or calculator to graph each side of the identities to show that each side yields the same graph.
e.g., graph $y = \sin(-\theta)$ and $y = -\sin\theta$.

EXTENSION

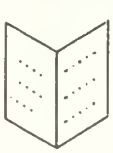
Several interesting approaches to the derivations of both the sum and difference formulas appear in textbooks and journals.

See Mathematics Teacher, Volume 83, Number 2, February 1990, pp. 84–86 and Mathematics Teacher, Volume 75, Number 1, January 1982, pp. 10–12 for interesting variations of the derivation of these formulas. Also refer to the prescribed resources for derivations of the sum and difference formulas.

Development of half arc and double arc formulas would also be appropriate extension topics.

As well, students could develop the sum and difference formulas for the tangent function. See Mathematics Teacher, Volume 82, Number 5, May 1989, pp. 384–385 for such a development.

RESOURCE CORRELATION



HM12:	pp. 320–324
MPP12:	pp. 221–226
M12:	pp. 236–240

CONCEPT 3

Students will be expected to demonstrate an understanding that trigonometric functions can be graphed on a Cartesian plane.

Students will be expected to:

3.1 graph the following forms of the sine, cosine and tangent functions:

$y = a \sin [b(\theta + \phi)] + d$

$y = a \cos [b(\theta + \phi)] + d$

$y = \tan \theta$

3.1.1 use electronic calculators or computers to draw and analyse the graphs of trigonometric functions

3.1.2 investigate the effects of the parameters a, b, ϕ, d on the graphs of the trigonometric functions using calculators or computers.

COMMENTS

The transformation of functions is a major topic in both Mathematics 20 and 33. It is important to help students recall the effects of a, c and d. Once the students understand the basic properties of the sine and cosine graphs the students could be asked to recall the various translation effects on the quadratic function. As each is discussed the students should be able to predict a similar effect on the sine and cosine graphs. These could be later confirmed with an actual graph.

The effect of b on transforming a graph is a new topic. Every attempt should be made to allow the students to discover this effect through the use of a graphing calculator or a computer.

Students should be able to write the equation if given the graph. Many students find this very difficult to do. The key is to focus the student on one period of the graph that preferably (but not necessarily) begins as close to origin as possible. Encourage them to draw a rectangle around this one period. From this rectangle it is relatively easy to identify the amplitude, phase shift, vertical translation and period. If they write the most general sine or cosine function:

$y =$ ____ sin _____ (x +/- _____) +/- _____

$y =$ ____ cos _____ (x +/- _____) +/- _____

the students should be able to fill in the missing values.

Students should also be able to produce a graph from a given equation. However, the vast majority of the graphs that the student will be asked to draw should be done using the available technology. For an excellent method of graphing the general sine function refer to the article by John Hornsby[1]. One caution is that his "B" is slightly different than the "b" referred to in this manual. Another interesting article[2] uses functions to make designs by limiting the domains. An example is given that produces a clown's face. Students could be challenged to produce designs of their own.

Many phenomena in nature can be reasonably represented with a sine or cosine function. Examples include tidal fluctuations, alternating electrical current, the height above the ground of a person on a ferris wheel or the length of the day as a function of the day of the year. (Use function notation whenever possible.) One approach to writing an equation that best represents the situation is to make a table of values. These could be graphed by hand or with a spreadsheet (that has a graphing capability). Most spreadsheets will easily allow production of a graph with discrete data points. The graphs can be presented in several ways but the most appropriate would be what is referred to as an x-y graph.

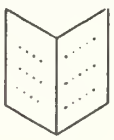
Note: Plotting graphs of this type (scatter plots) is not possible with all graphing calculators and it is never as easy as with a spreadsheet.

The transformation concept is not required for the tangent graph (i.e., the student needs only to graph $y = \tan x$). However, the student should understand what happens as values for the angle approach places where the tangent is undefined (i.e., any integral multiple of 90°). Although the concept of asymptotes is not discussed in any detail in Mathematics 30, it may be useful to use the terminology here and to relate this to the study of the hyperbola in the Quadratic Relations unit.

TECHNOLOGY INTEGRATION

Computer Graphing Experiments 2 (Addison-Wesley) has a number of activities and worksheets that could be used in conjunction with any graphing technology.

RESOURCE CORRELATION



HM12:	pp. 296–307
MPP12:	pp. 191–211
M12:	pp. 178–225

1. Hornsby, E. John Jr. "A Method of Graphing f(x) = Asin (Bx + C) + D." Mathematics Teacher, January 1990, p. 51.

2. Iacocca, Joan M. "Trigonometry and Design," Mathematics Teacher, October 1986, p. 526.

CONCEPT 4

Students will be expected to demonstrate an understanding of the methods used to solve trigonometric equations.

Students will be expected to:

4.1 solve first and second degree trigonometric equations including those of multiple angles in the domain $0 \leq \theta \leq 2\pi$

4.1.1 use electronic calculators or computers to solve trigonometric equations by evaluating the graphs of trigonometric functions

4.2 demonstrate the relationship between the solution of a trigonometric equation and the graph of the corresponding function.

COMMENTS

In previous mathematics courses, students have solved equations like $\sin\theta = 0.1234$ where $\theta = 7.1^0$. Although students are familiar with reference or related angles, they may not have applied this to the above equation (i.e., they may not have studied how to determine **two** values of θ for which $\sin\theta = 0.1234$).

Most equations can also be solved as a system of equations.

For example, $2\sin\theta - 1 = 0$ could be solved as follows:

$$2\sin\theta - 1 = 0$$
$$\sin\theta = 1/2$$

Now have the students consider the system:

1 $\quad y = \sin\theta$

2 $\quad y = 1/2$

The students now have the opportunity to investigate the intersections of the graph of $\sin\theta$ and the horizontal line $y = 1/2$. This investigation may allow them to understand that the solutions are the points of intersection (i.e., $\pi/6$ and $5\pi/6$).

In previous mathematics courses, students have studied the solution of first and second degree equations. In solving trigonometric equations, they have the opportunity to use a variety of the techniques they have used previously.

Students should be able to write solutions using both radian and degree measures. In addition, they should be able to use exact values (i.e., multiples of $\pi/4$, $\pi/6$ or $30°$, $45°$), as well as those angles that are not multiples (like $32°$, $158°$, $\pi/5$, 1.23 rad).

PROCESS/PROBLEM-SOLVING CONTEXT

Problem: At Lethbridge the time, in hours, of the sunrise for any day as a function of the day of the year is given by the formula

$$t(d) = -1.8 \sin 2\pi \frac{(d-78)}{365} + 6.3$$

Note: Calculator should be set on "radians."

According to this formula, when did the sun rise on April 10 (day 100)?
Answer: on day 100, the time would be 5.6345 h or 05:38 h.

How could the formula be rewritten to use degree measure? Also, determine when (as in the date), the sun will rise at 06:30h.
Answer: approximately day 84 (March 25) and day -98 (Sept. 24).

TECHNOLOGY INTEGRATION

The view of a trigonometric equation as a system outlined above can be reinforced using graphing calculators or computers. In addition, a trial-and-error approach using a spreadsheet can serve to estimate, or in some cases, actually solve the equations.

Calculators will be required in those situations where the answers are not multiples of $\pi/6$ or $\pi/4$. Along with the concepts of related or reference angles studied in Mathematics 20/23 and the "CAST" rule, calculators can be used to solve most trigonometric equations. Students should be made aware of the decimal equivalents of "exact" radian or degree measures so that they can identify them from their calculators (for example $\sqrt{3}/2 = 0.8660\ldots$, $\sqrt{2}/2 = 0.7071\ldots$, $\sqrt{3} = 1.732\ldots$).

Problem: Solve: $4\sin^2\theta = 3$ for $0 \leq \theta < 2\pi$
Solution: $\sin^2\theta = 3/4$
$\sin\theta = \pm 0.8660\ldots$ which is the approximate decimal equivalent for $\sqrt{3}/2$. Therefore, the answers are $\pi/3$, $2\pi/3$, $4\pi/3$, $5\pi/3$.

EXTENSION

This topic can easily be extended into equations that involve two functions that rely for their solution on the use of identities. Examples of such questions are in the following form Holtmath 12:

$$\frac{\cos(90^\circ-\theta)}{1\cos\theta}=\sqrt{3} \qquad \text{and} \qquad \sin\theta=1-\cos 2\theta.$$

SOLVING EQUATIONS OF THE TYPE
$a\cos u+b\sin u=c$

Three often used methods for solving

(1) $a\cos u+b\sin u=c$

are as follows:

- Express (1) in the form $R\cos(u-v)=c$ with

$$R=\sqrt{a^2+b^2}$$

and v given by $\tan v=b/a$. Solve $R\cos(u-v)=c$.

- Use $t=\tan u/2$; then

$$\cos u=\frac{1-t^2}{1+t^2}$$

and

$$\sin u=\frac{2t}{1+t^2}$$

which reduces (1) to the quadratic equation

$$a(1-t^2)+b(2t)=c(1+t^2).$$

Solve for t to obtain u/2 and finally u.

- Draw the graph of $a\cos u+b\sin u=y$ and find the intersection with $y=c$.

Another approach is different from these methods and is adaptable to quite a different graphical method. To solve (1), let $\cos u=x$, $\sin u=y$. Then

(i) $$x^2+y^2=1.$$

and the given equation becomes

(ii) $$ax+by=c.$$

Solve (i) and (ii) as a pair of simultaneous equations to obtain solution pairs (x, y). Determine u from these pairs.

Example:

$$\sqrt{3}\cos u + \sin u = 1$$

yields

$$x^2 + y^2 = 1$$

and

$$\sqrt{3}x + y = 1,$$

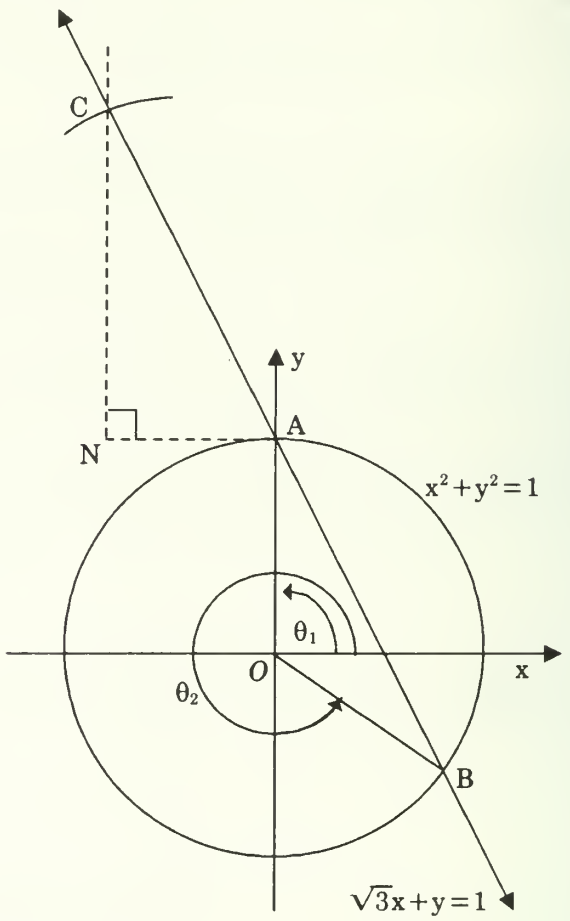


Fig. 1 θ_1 and θ_2 are solutions

θ_1

so

$$x = \frac{1-y}{\sqrt{3}}$$

Therefore

$$\frac{(1-y)^2}{3} + y^2 = 1,$$

or

$$1 - 2y + y^2 + 3y^2 = 3,$$

or

$$4y^2 - 2y - 2 = 0,$$

giving $y = 1$ or $-1/2$ with corresponding values of x being $x = 0$ or $x = \sqrt{3}/2$. So cos $u = 0$ and sin $u = 1$, yielding $u = 90°$; or cos $u = \sqrt{3}/2$ and sin $u = -1/2$, yielding $u = 330°$. The equations (i) and (ii) can be solved graphically by drawing the unit circle $x^2 + y^2 = 1$ and the straight line $ax + by = c$ and reading off the solutions. See Figure 1.

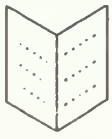
The circle $x^2 + y^2 = 1$ is easy to draw. One point on the line $\sqrt{3}x + y = 1$ is A = (0, 1). Another point C is obtained through constructing the ΔANC with $AN = 1$, m $\angle ANC = 90°$ and $AC = 2$, giving $CN = \sqrt{3}$ and slope $-\sqrt{3}$.

The solution is angles θ_1, θ_2 made respectively by radii $\overline{OA}$ and $\overline{OB}$ with the positive direction of the x-axis.

Note: The straight line $ax + by = c$ requires the use of either two points on the line or one point and the slope. Where a, b, and c are rational numbers, the points are obtained with good accuracy. When, as in the example, c is rational, at least one of a and b is rational, and the other is a radical of the form $\sqrt{p}, p \in N$, good accuracy is obtained using construction similar to the one shown for the line $\sqrt{3}x + y = 1$.

Reprinted from National Council of Teachers of Mathematics, Mathematics Teacher, April 1989, pp. 263-264.

RESOURCE CORRELATION



HM12:	pp. 327–329
MPP12:	pp. 230–234
M12:	pp. 251–255

STATISTICS
SUGGESTED PROGRAM EMPHASIS – 15%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that a bivariate distribution involves two variables that may have some relationship to each other.	Students will be expected to plot sets of bivariate data on a scatter plot. Students will be expected to plot a line of best fit on a scatter plot using the median fit method. Students will be expected to develop and use prediction equations of the line of best fit to make inferences for populations. Students will be expected to determine the apparent correlation between the variables of a bivariate distribution from a scatter plot, including the strength and direction of the correlation. Students will be expected to collect, organize and analyze sets of bivariate data.	Students will be expected to apply statistical processes and statistical reasoning in investigations involving bivariate data.
Students will be expected to demonstrate an understanding that data can be distributed normally, and that a normal distribution has particular characteristics that can be used to describe and analyze many situation.	Students will be expected to find and interpret the mean and standard deviation of a set of normally distributed data. Students will be expected to apply the characteristics of a normal distribution. Students will be expected to find and apply the standard normal curve and the z-scores of data that are normally distributed.	Students will be expected to use calculators or computers to calculate the mean and standard deviation of sets of normally distributed data. Students will be expected to solve problems involving data that are normally distributed. Students will be expected to apply z-scores to solve problems involving probability distributions.
Students will be expected to demonstrate an understanding that the results of a survey can be interpreted with measurable degrees of confidence.	Students will be expected to distinguish between a population and a sample and assess the strengths, weaknesses and biases of given samples. Students will be expected to collect and organize the results of yes/no surveys taken from defined samples. Students will be expected to draw box plots of the results of multiple samples. Students will be expected to use charts of 90% box plots to find the confidence interval within which survey results can be interpreted. Students will be expected to draw statistical conclusions, make inferences to populations and explain the confidence with which such conclusions and inferences are made based on the results of yes/no surveys.	Students will be expected to design and administer a simple survey. Students will be expected to collect and organize the results of a simple survey. Students will be expected to carry out investigations involving multiple samples taken from populations with known and unknown proportions of yeses. Students will be expected to use statistical inferences to solve problems. Students will be expected to design and administer a survey to a random sample of a population, collect and organize the responses, and analyze the results, including making inferences to the population and evaluating the results for the confidence with which they may be held.

CONCEPT 1

Students will be expected to demonstrate an understanding that a bivariate distribution involves two variables that may have some relationship to each other.

Students will be expected to:

1.1 plot sets of bivariate data on a scatter plot
1.2 plot a line of best fit on a scatter plot using the median fit method
1.3 develop and use prediction equations of the line of best fit to make inferences for populations
1.4 determine the apparent correlation between the variables of a bivariate distribution from a scatter plot, including the strength and direction of the correlation
1.5 collect, organize and analyze sets of bivariate data
    1.5.1 apply statistical processes and statistical reasoning in investigations involving bivariate data.

COMMENTS

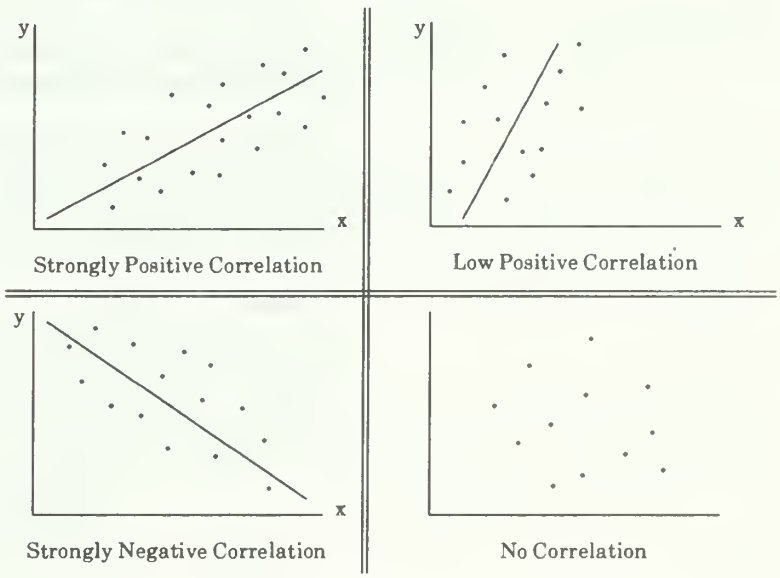
Students have worked with univariate data in Mathematics 10. These were situations where one variable is measured for a given group, for example, the heights of students in a class. The techniques and skills used include the representation of data by line plots, stem and leaf plots, box plots and calculating various averages. This concept involves situations where two variables are measured for each individual in a group; for example, the height and circumference of the neck are measured for each class member. The data collected for each student can be considered a point on a Cartesian plane. The resulting graph is called a scatter plot. The scatter plot is a useful technique for investigating the relationship between two variables. Some relationships may be approximated with a "best fit" line, in which case the equation of the line should be found. The line and equation can be used to make predictions. The limitations of such a procedure should be mentioned especially when the data are extrapolated.

There are different ways to plot a line of best fit. The common method described in texts is the median method. Many calculators will calculate the slope and y-intercept of the best line using the "least squares" method. Some spreadsheets will do this type of calculation as well. It is important to remember that calculating numbers is not the focus of this unit. Students should be able to determine the equation of this line once they know the slope and y-intercept. Also important to note is that a straight line may not be appropriate for all types of data. The relationship may be curvilinear. The data could be clustered into two groups where it may be appropriate to draw a different straight line for each cluster.[1]

1. Landwehr, James, Exploring Data: Quantitative Literacy Series, 1986, p. 133.

The strength and direction of the relationship between the variables can be summarized with a number called the correlation coefficient. If all the points fall perfectly along a line with positive slope then the correlation coefficient is $+1$. If the points exactly form a line of negative slope then the correlation coefficient is -1. A value of zero for the correlation coefficient means that no relationship exists between the variables. In most cases the correlation will be a value between -1 and $+1$. A positive coefficient indicates a positive relationship, meaning that as one variable increases, so does the other. A negative correlation coefficient indicates a negative relationship, where one variable increases as the other decreases. Students should not be required to calculate the coefficient unless software exists that does this easily. A qualitative understanding of this value is much more significant.

Correlation in no way implies causality. There are many reasons why two variables may correlate highly. For example, you may find a positive correlation between the amount of math homework that a student does and their positive feelings toward math. It may not make sense to assume, however, that more homework will give a student more positive feelings toward the subject. For a more extensive discussion of this topic see Campbell's book Flaws and Fallacies.[2]



2. Campbell, Stephen, Flaws and Fallacies in Statistical Thinking, Prentice Hall, 1974, p. 152.

The NCTM Standards document[3] suggests a number of things that are applicable to this unit. Student activities should focus on asking questions about their environment. Examples given should involve real, not contrived, data. Projects must be an integral part of the learning and evaluation process. The focus must **not** be on the ability to calculate a particular value or values, but rather on analyzing and communicating numbers and ideas.

An excellent way to begin study of this concept is to ask students to suggest variables that might be related in positive and negative ways. A positive relationship might exist between hours of study and marks on tests or the height of a father and his son. A negative relationship might exist between the number of hours of "after-school" employment and marks in school. Students should be given an opportunity to investigate these relationships by collecting data and making scatter plots.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have students look in almanacs or statistical yearbooks and collect information about various countries. From the abundance of data they may have a theory about two variables that are related (positively or negatively).

e.g., rainfall and GNP
population and pollution
area and population

Investigate this relationship and prepare a presentation. This would also be an excellent cooperative project.

Activity 2: Have students ask the principal for attendance figures on various days. Phone the weather office and ask for the average temperature on those days. Do they correlate? Can you predict the attendance if you know the temperature?

Activity 3: Sports fans have access to an abundance of statistics. Every day the sports section of the newspaper is filled with opportunities for investigations of relationships between variables. Statistical yearbooks that summarize last years' results are available on newsstands.

e.g., hockey – shorthanded goals scored: for and against
goals scored: for and against
points in the standings and penalty minutes
football – yards of offence and points scored
baseball – runs scored and number of hits

3. National Council of Teachers of Mathematics. Curriculum and Evaluation Standards for School Mathematics, Preston, VA, p. 105.

Activity 4: From the beginning of the year ask students to keep an accurate log of the amount of work (in hours) they do in each unit. Attempt to correlate with their marks in these units.

Activity 5: Present the students with a statistical argument that will predict their final mark in the course. For example, by keeping records of students' performance in Mathematics 20 and Mathematics 30 from previous years, the current students could predict their final mark in the course. Have the students critique this argument.

Activity 6: Get students to collect univariate data on each member of the class. It is important to keep track of the student's name with each value collected. A wide variety of data is best.
Examples:
- the mark on a spelling test designed by the experimenter
- the number of free throws out of 10 using a nerf basketball hoop
- how many of the Seven Dwarfs' names can be recalled
- the accuracy of a guess about the number of jelly beans in a jar
- the number of albums, CDs or cassettes owned.

Each student should summarize their findings using the techniques for univariate data. The data and summaries could be posted around the room. Students are then asked to investigate two possible relationships using these data:

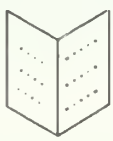
1. A relationship that they believe has a highly positive or highly negative correlation coefficient (i.e. close to either $+1$ or -1). For example the mark on the spelling test and the number of "Seven Dwarfs" recalled may correlate highly.

2. A relationship that they believe has a very small correlation coefficient (i.e., close to zero).

TECHNOLOGY INTEGRATION

Scientific calculators and spreadsheets are designed specifically to handle the calculation of correlation coefficients and equations of "best-fit" lines. Graphing calculators and some spreadsheets will easily produce a scatter plot of the data.

RESOURCE CORRELATION



CONCEPT 2

Students will be expected to demonstrate an understanding that data can be distributed normally, and that a normal distribution has particular characteristics that can be used to describe and analyze many situations.

Students will be expected to:

2.1 find and interpret the mean and standard deviation of a set of normally distributed data
- 2.1.1 use calculators or computers to calculate the mean and standard deviation of sets of normally distributed data

2.2 apply the characteristics of a normal distribution
- 2.2.1 solve problems involving data that are normally distributed

2.3 find and apply the standard normal curve and the z-scores of data that are normally distributed
- 2.3.1 apply z-scores to solve problems involving probability distributions.

COMMENTS

Many phenomena exhibit characteristics that are normally distributed. The heights, weights, I.Q. and reaction times of people are just a few examples. Repeated measurements of a single quantity will give values that are normally distributed (see Activity One).

Two distributions may have the same mean but one may be "flatter" than the other. This variability of dispersion could be shown by comparing results such as those in Activity 1. The problem becomes how to measure dispersion (see Activity 2). The range and standard deviation are common measures of dispersion. Standard deviation is often a difficult concept initially, but getting students to estimate it before using their calculator is often useful (see Activity 3). Students should be shown how to calculate the standard deviation using the formula but the emphasis should be on using calculators that have statistical functions.

Gallup polls, light bulb manufacturers, car builders and Neilsen ratings all use samples to draw conclusions about a much larger population. The mean and standard deviation of random samples are closely related to those of the population from which they are taken. The sample mean ($\overline{X}$) is usually assumed to be a good approximation of the population mean (μ). Most calculators distinguish the sample and population standard deviation (s and σ respectively). If a sample is being used to estimate the standard deviation of a population, the sample standard deviation is used. If the sample is actually the entire population, use the population standard deviation (see Activity 4).

The standard normal curve and associated tables of probabilities have many applications. These are very well illustrated in the prescribed textbooks.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have each student measure their reaction time using a metre stick. The student holds his hand over the edge of the desk and catches a ruler that is dropped by another student. The distance of the drop is recorded. Record about 20 measurements for each individual. Try to ensure that each "experiment" is identical. A stem and leaf plot should reveal an approximately "bell-shaped" curve, which is characteristic of a Normal Distribution. Record the mean for each student and repeat the stem and leaf plot for the class as a whole.

Note: Physics students should be able to convert the distance into a time by using a calculator and the formula below. However the distances are actually "nicer" to work with.

$s = 0.5 \times 980 \times t^2$ where s is the distance in centimetres, 980 is the acceleration due to gravity in cm/s^2 and t is the reaction time in seconds.

Activity 2: For each set of data calculate the mean, median and range.

#1→ 10 20 30 40 50 60 70 80 90 100

#2→ 10 43 46 49 52 58 61 64 67 100

Compare and contrast the two sets of data.

Note: The student should feel the importance of having some measure of dispersion other than the range after this activity.

Activity 3: As a rough guide the mean plus or minus the standard deviation should include **approximately** two thirds of the data collected. Estimate the standard deviation of the data below.

14 15 18 20 21 23 24 25 25 26 26 27 30 31 32 32 33 35

Answer: The mean is about 25.4 Two-thirds of the data would be a total of 12 numbers out of the 18. This means that we should include about six numbers on either "side" of 25.4 – namely the numbers 20 through 32. One estimate is to take half the difference between these numbers, which gives $(32-20)/2$, or 6. The actual answer will often vary widely from this estimate, but this "rule" can be a guide to understanding the concept and useful to detect major errors in "keying" numbers into the calculator.

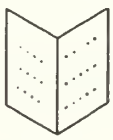
Activity 4: Cut out each of the 200 squares in Figure 1. The numbers represent a population with a mean of 50 and a standard deviation of 10. Take random samples of 10 numbers and calculate the mean and standard deviation of the samples. Compare these to the population figures.

Note: It may be better to laminate the pages before cutting the squares because they tend to stick together.

Problem: If you wanted really hot weather for your holiday would you rather visit country A or country B? Explain your reasoning.

COUNTRY	AVERAGE TEMPERATURE	STANDARD DEVIATION OF TEMPERATURE
A	22°C	5°C
B	25°C	8°C

RESOURCE CORRELATION



AW: pp. 477–498
Nelson: pp. 449–460
Holt: pp. 393–403

ADDITIONAL RESOURCES

Against All Odds: Inside Statistics

a. Normal Distributions (program 4)
b. Normal Calculations (program 5)

This video series is available from ACCESS and shows many applications of the Normal Distribution. They show real men and women actually using mathematics in their jobs!

Mathematics 30 – Statistics – Figure 1

27	28	29	30	31	32	32	33	33	34	34		35	35	35
36	36	36	37	37	37	38	38	38	38	39	39	39	39	39
40	40	40	40	40	41	41	41	41	41	42	42	42	42	42
42	43	43	43	43	43	43	44	44	44	44	44	44	44	45
45	45	45	45	45	45	46	46	46	46	46	46	46	46	47
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56	56	56	57	57	57	57	57	57	58	58	58	58	58	58
59	59	59	59	59	60	60	60	60	60	61	61	61	61	62
62	62	62	63	63	63	64	64	64	65	65	65	66	66	67
67	68	68	69	70	71	72	73							

CONCEPT 3

Students will be expected to demonstrate an understanding that the results of a survey can be interpreted with measurable degrees of confidence.

Students will be expected to:

3.1 distinguish between a population and a sample and assess the strengths, weaknesses and biases of given samples

3.2 collect and organize the results of yes/no surveys taken from defined samples
- 3.2.1 design and administer a simple survey
- 3.2.2 collect and organize the results of a simple survey

3.3 draw box plots of the results of multiple samples
- 3.3.1 carry out investigations involving multiple samples taken from populations with known and unknown proportions of yeses

3.4 use charts of 90% box plots to find the confidence interval within which a survey result can be interpreted
- 3.4.1 use statistical inferences to solve problems

3.5 draw statistical conclusions, make inferences to populations and explain the confidence with which such conclusions and inferences are made based on the results of yes/no surveys
- 3.5.1 design and administer a survey to a random sample of a population, collect and organize the responses, and analyze the results, including making inferences to the population and evaluating the results for the confidence with which they may be held.

COMMENTS

This concept involves making and interpreting surveys. An appropriate introduction would be to show students examples of surveys from the media. There are various types of surveys but this concept deals only with "Yes–No" surveys.

In this type each person asked must respond with "Yes" or "No."

Examples of acceptable "Yes–No" survey questions would be:
a. Did you have an average over 80% last year?
b. Do you like rap music?

Examples of questions that would be unacceptable for "Yes–No" surveys would be:
a. What was your average last year?
b. React to the statement below by circling a response: "I love rap music."

strongly agree	agree	neutral	disagree	strongly agree

Surveys are an integral part of inferential statistics. The goal is to infer something, usually the mean, of a large group (called the population) by collecting information from a subset of this group which is called a sample. One of the problems is that a random sample does not always accurately reflect the population (see Activity 1).

A survey with a sample size of 20 could select eight smokers at random. How accurate is the conclusion of this survey that 40% of the population are smokers? (i.e. 8 out of 20 =40%) If the surveyor were to say that the mean is 40% + − 10% in nine out of 10 cases, someone reading the survey would have a better indication of the limitations of the information. This "additional" information is called the 90% confidence interval of the mean. It reveals that if the survey were conducted 10 times then nine out of 10 times the mean number of smokers in the survey would be between six and 10. (i.e. between 30% and 50% if the sample size is 20 people). One of the major problems to be tackled in this concept is the establishment of confidence intervals.

The establishment of confidence intervals requires some backtracking. The focus will be on a population that has a known number of "yeses". In a room filled with 40 smokers and 60 nonsmokers, a random selection of 10 names from a hat will not always give four smokers and six nonsmokers. A simulation is an effective method to begin the discussion of confidence intervals.

A random number table can be used to simulate taking samples of 10 from a known population of 40% smokers. If the digits 0, 1, 2 and 3 represent "yeses" and all others represent nonsmokers then many samples can be taken in a short time using the table. Several textbooks also list computer programs that will do thousands of samples in minutes.

Here is an example of a computer simulation.

Population percentage of yeses = 40%
Sample size = 10
Number of samples analyzed = 100

number of yeses	frequency	sample proportion	likely or unlikely
0	3	0	unlikely
1	45	.10	unlikely
2	115	.20	likely
3	224	.30	likely
4	240	.40	likely
5	195	.50	likely
6	129	.60	likely
7	35	.70	unlikely
8	13	.80	unlikely
9	1	.90	unlikely
10	0	1.00	unlikely

Notes: * Sample proportion $= \frac{\text{no. of yeses}}{\text{sample size}}$

* The frequency of each class can be predicted using the Binomial Distribution formula because the "Yes-No" Survey is a binomial experiment. Students need not ever use this formula but it may be useful for the instructor to check the simulations.

$$\text{frequency} = N \times \frac{n!(p^y) \times (1-p)^{n-y}}{y!(n-y)!}$$

where N = number of samples taken, n is the size of each sample, y is the number of yeses in the sample and p is the population percentage of yeses.

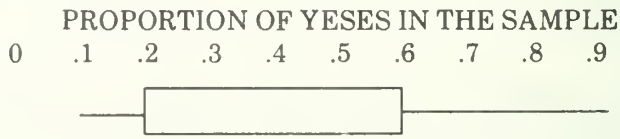
e.g., For our simulation $N = 1000$, $n = 10$, $p = .40$ and an arbitrary choice of y is 7. Therefore:

$$\text{frequency} = \frac{1000 \times 10!(.40^7)(.60)^3}{7!3!} = 42.46$$

* Tables are readily available for the binomial distribution. Most will list the probability of a certain number of yeses within each sample size.

The curriculum focuses on 90% confidence intervals. To do this, 90% of the values (900 out of 1000) are labelled as likely to occur. At either end, 5% of the values are labelled as unlikely numbers of yeses. The total frequency of these values is '3+45+35+13+1+0' or 98. This represents about 10% of the simulations that were done. The total frequency of the other values is 902 which is about 905 of the simulations that are done. Therefore for a population of 40% yeses and a sample size of 10 you would expect to find the number of yeses in a sample to be between two and six about 90% of the time.

A box plot can be used to summarize this table. The box would contain 90% of the values and the whiskers 5% each. These are called 90% box plots. The whiskers should never contain more than 5% of the values. The sample proportions that are likely are therefore inside or on the borders of the box. For the simulation here the likely proportions are between 0.2 and 0.6 using 90% confidence intervals.



The textbooks all contain tables of 90% box plots for various sample sizes. These box plots, like the one above, are for populations that vary from 0% to 100% yeses. A class project could be to build one of these tables.

Students' previous experience with box plots will be with ones that use the median and are therefore 50% box plots. This means that the box contains 50% of the values while each whisker has 25% of the values.

This chart of 90% box plots is used to establish 90% confidence intervals for surveys. If, for example, you conduct a survey with a sample size of 20, you would refer to the chart with that size. If you recorded eight "yeses" in your sample of 20 then your sample proportion is 40%. However, you want to know the population percentage of "yeses". A sample proportion of 40% yeses is likely to occur in a wide variety of populations. By finding .4 on the top row of the chart and drawing a vertical line it can be seen that .4 is a likely sample proportion in populations that have from 25% to 60% "yeses". The results of the survey, therefore, should be reported as 40% $\pm$ 15% in nine out of 10 cases.

A major portion of the evaluation of this concept should be the actual design and evaluation of a survey. The focus should be on the students' environment and answering questions that are of interest to them. Every effort should be made to ensure a truly representative sample and to be aware of any bias that might enter into their results.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Introduce the idea of surveys and their terminology by asking a question of everyone in the class. The question should be of the "yes-no" variety.

a. Do you like watching NFL football on TV?
b. Did you do more than one hour of homework last night?
c. Does your family have a CD player?

Calculate the percentage of yeses for the class. Now randomly select five students and get their answers to the same question and calculate the mean for this sample of five. Compare the population and sample means. Repeat with several other samples.

Activity 2: Have students collect examples of surveys that are discussed in the paper or magazines. Note which ones are "yes-no" surveys and the results of the survey, especially the confidence interval.

Activity 3: Have the class cooperatively construct a 90% box plot with sample sizes of 20. Each group of two could work on a different population percentage of yeses using a random sample. These could be constructed using a variety of techniques including a random number generator included on some calculators, random number tables, computer programs or even specially designed spinners. A spinner for a 40% yes population, for example, could feature a circle divided into 10 pieces where four pieces are labelled "yes" and six pieces labelled "no".

Activity 4: Ask students to construct a 95% box plot for samples of 20 and the population percentage of yeses as 40%. Compare with the 90% box plot. The simulations done previously can still be used. In simulations using 1000 trials, approximately 50 trials will be labelled as unlikely and the "middle" 950 as likely. To be 95% confident of the answer the range of possible values must be widened.

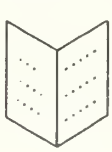
Other Activities: For a collection of activities, ideas and teaching strategies see the book Exploring Surveys and Information from Samples.[1]

TECHNOLOGY INTEGRATION

As noted above, there are many opportunities to use computers and calculators in this unit. Some spreadsheets even have a random number generator. These can be used to do simulations of various populations.

1. Landwehr, James et al., Exploring Surveys and Information from Samples. Dale Seymour Publications, Palo Alto, CA, 1987.

RESOURCE CORRELATION



QUADRATIC RELATIONS
SUGGESTED PROGRAM EMPHASIS – 10%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that a locus is a system of points that satisfies a given condition.	Students will be expected to write the equation of a simple locus given a condition. Students will be expected to sketch a simple locus given a condition.	
Students will be expected to demonstrate an understanding that a circle is the locus of all points equidistant from a fixed point.	Students will be expected to describe the effects of the parameters h, k and r in the standard form of the equation of a circle $(x-h)^2+y-k)^2=r^2$. Students will be expected to determine the equation of a circle in standard form given the information necessary to find any suitable combination of the following: coordinates of the centre, the length of the radius and the coordinates of points on the circumference. Students will be expected to change the form of the equation of a circle from standard form to general form, $x^2 + y^2 + Dx + Ey + F = 0$ and vice versa. Students will be expected to sketch the graphs of circles given the equation in standard or general form or information necessary to find any suitable combination of the following: coordinates of the centre, the length of the radius and the coordinates of points on the circumference.	Students will be expected to solve problems which involve finding and analyzing the equation of a circle and finding and analyzing the characteristics of a circle or a circular region. Students will be expected to investigate the effects of the numerical coefficients in the general form of the equation of a circle on the graph of the circle. Students will be expected to draw and analyze the graphs of circles using calculators or computers.
Students will be expected to demonstrate an understanding that a parabola is the locus of all points equidistant from a fixed point (focus) and a fixed line (directrix).	Students will be expected to verify that standard form of the equation of a parabola $(y-k)^2=4p(x-h)$, $(x-h)^2=4p(y-k)$ can be derived from the locus definition. Students will be expected to describe the effects of the parameters h, k and p in the standard form of the equation of the parabola.	Students will be expected to investigate how the parameters h, k and p affect the location, orientation, size and shape of the parabola.

CONCEPT 1

Students will be expected to demonstrate an understanding that a locus is a system of points that satisfies a given condition.

Students will be expected to:

1.1 write the equation of a simple locus given a condition

1.2 sketch a simple locus given a condition.

COMMENTS

Locus definitions are concise English definitions that clearly outline the geometric conditions to be met by the locus. They provide a visual image that students can understand and clues about how to define the locus algebraically. These characteristics are extremely useful as students begin to look at quadratic relations.

The locus definitions of parabola, ellipse and hyperbola are fairly difficult for students to understand without some preliminary practice. It is for this reason that work with simpler loci should precede the work with the four quadratic relations.

Particular attention should be paid to the idea of a condition that describes the relationship between points (the locus) and some given characteristic. A locus may be a line or lines, a curve or curves or a particular shape. Care should be taken that the conditions given and the loci expected at this level are not beyond the students' ability to draw, describe and understand. Introductory questions based on material from previous courses can be used, such as drawing the locus and finding the equations of all points that pass through a given point and that have a given slope.

Keep in mind the overall objectives of this unit. Except for those who have demonstrated strong background skills, questions should be limited to those that address the specific objectives. Students should become competent in giving a complete description of each conic section in the geometric and algebraic format.

It is an advantage to students that the first step in working with a locus is to draw it. This can then be followed by finding the equation. However, equations can be found only if the locus is drawn on a Cartesian Plane. In those cases, finding the equation may be aided by using known coordinate geometry formulas such as those for distance, midpoint and slope.

PROCESS/PROBLEM-SOLVING CONTEXT

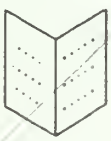
Activity: Have students draw pictures of some simple loci, such as the following: all points equidistant from a pair of parallel lines, all points equidistant from two fixed points, all points equidistant from a pair of intersecting lines, all points equidistant from a fixed point, all points equidistant from a line, all points equidistant from and between the arms of an angle. Have them compare their loci and defend their decisions. Note that in some cases, the locus will be complete only if it has two parts. Note that the distance to a line is always measured perpendicularly.

Activity: Give students diagrams of some simple loci and have them write conditions to describe them.

Activity: Have students draw and write conditions for their own loci.

Problem: Draw the locus and find the equation of all points that are twice as far from (5, 4) as they are from (−1, 4). (For a thorough look at a similar problem, see Mathematics Teacher, Volume 82, Number 2, February 1989.

RESOURCE CORRELATION



HM12:	pp. 153
MPP12:	pp. 298–302
M12:	pp. 91–94

CONCEPT 2

Students will be expected to demonstrate an understanding that a circle is the locus of all points equidistant from a fixed point.

Students will be expected to:

2.1 describe the effects of the parameters h, k and r in the standard form of the equation of a circle $(x-h)^2+(y-k)^2=r^2$

2.2 determine the equation of a circle in standard form given the information necessary to find any suitable combination of the following: coordinates of the centre, the length of the radius and the coordinates of points on the circumference

2.2.1 solve problems that involve finding and analyzing the equation of a circle and finding and analyzing the characteristics of a circle or a circular region

2.3 change the form of the equation of a circle from standard form to general form, $x^2 + y^2 + Dx + Ey + F = 0$ and vice versa

2.3.1 investigate the effects of the numerical coefficients in the general form of the equation of a circle on the graph of the circle

2.4 sketch the graphs of circles given the equation in standard or general form or information necessary to find any suitable combination of the following: coordinates of the centre, the length of the radius and the coordinates of points on the circumference

2.4.1 draw and analyze the graphs of circles using calculators or computers.

COMMENTS

The students should be given the locus definition of a circle and attempt to find the equation of the locus for specific examples. They should graph the locus before this is attempted. After several examples the student will probably see the connection between the graph, equation, centre and radius and no longer need to go back to the locus definition to write the equation.

Some students have trouble with arbitrary nature of the symbols h, k and r. It may be important to mention that the standard form of the circle could just as easily be $(x-d)^2 + (y-f)^2 = r^2$ or any other convenient letters.

As the students discuss the effects of changing h, k and r they need to be reminded of the work done on transformation of functions (i.e. replacing x with $(x-h)$ or y with $(y-k)$ has the effect of shifting the graph).

Most students will need to be reminded of the difference between a function and a relation. The circle is a quadratic relation.

Most graphing calculators require that the relation be solved for y in order to graph. When this is done for a circle, a positive and negative of a square root is the result. This problem can be dealt with by graphing each separately and overlaying the graphs. Also note that on some graphing calculators the resulting graph looks somewhat elliptical because the screen is rectangular. By adjusting the scales on the axes, it may be possible to correct the distortion.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem: If the function $y=x^2$ is transformed by multiplying by two, it gives the new function $y=2x^2$, which has a different graph. Use a graphing calculator to see the differences. What happens to the circle $x^2+y^2=9$ when the equation is multiplied by two to become $2x^2+2y^2=18$? Again look for the differences with a graphing calculator. Explain the result.

Problem: The general form of the equation of a circle $x^2+y^2+Dx+Ey+F=0$. Find the values of D, E and F for the circle with a centre $(-1, 4)$ and radius 5. If the values of D and E are fixed at these values but the value of F is allowed to vary, describe the effect on the graph of the equation. What range of values can F have?

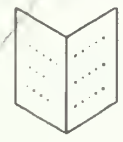
Problem: Find the equation of the circle that passes through the points M(2, 3), N(1, −4) and P(5, 4). Try to encourage a number of approaches. First graph the perpendicular bisectors of two possible chords either by hand or with a graphing program without finding the equation of either. Second, give the centre the label C(h, k). The distance MC = distance NC and MC = PC. Solve the system of two equations and two unknowns. A third approach is to solve the system of equations of the two perpendicular bisectors for the centre. There are other approaches as well.

Problem: Find the equation(s) of a circle (circles) passing through the points $(-2, 7), (5, 4)$ with a radius of 6.

Problem: Find the points of intersection of the circle with radius 6 and centre (2, 3) with the line $y=2x-1$. Is there a line parallel to this line that has only one point of intersection with the given circle? If so, find the equation of the line and the point of intersection.

Problem: Find the equation of the line that is tangent to both $(x-1)^2 + (y-3)^2 = 4$ and $(x+4)^2 + (y+5)^2 = 9$.

RESOURCE CORRELATION



HM12:	pp. 154–156
MPP12:	pp. 303–310
M12:	pp. 95–99; 103–108; 115–122; suppl. p. 1–5

CONCEPT 3

Students will be expected to demonstrate an understanding that a parabola is the locus of all points equidistant from a fixed point (focus) and a fixed line (directrix).

Students will be expected to:

3.1 verify that the standard form of the equation of a parabola $(y-k)^2=4p(x-h)$, $(x-h)^2=4p(y-k)$ can be derived from the locus definition

3.2 describe the effects of the parameters h, k and p in the standard form of the equation of the parabola

3.2.1 investigate how the parameters h, k and p affect the location, orientation, size and shape of the parabola.

COMMENTS

There are a few points to consider when preparing for this section. The first is that the students have studied parabolas before. This could be, however, as much a hindrance as a help because the form of the equation is different and the concepts are developed differently. The student should understand the relationship of earlier study to this unit. The study of the parabola in Mathematics 20 was restricted to the quadratic function. In Mathematics 30, study is extended to include quadratic relations. A more general approach is taken to encompass the different orientations.

A second point to note is that graphing calculators and computers are encouraged as aids to understanding parabolas. Some calculators and software, however, are inadequate because they don't graph relations as well as they graph functions. Therefore, students may have to work with the form $(x-h)^2 = 4p(y-k)$ and generalize their results to the form $(y-k)^2 = 4p(x-h)$ (or become inventive with their use of the technology)!

Next teachers should note that the only standard forms defined are those for vertex at (h, k). This implies that the standard forms $y^2=4px$ and $x^2=4py$ should be considered as instances of the standard forms rather than as additional standard forms that are distinct. Finally, teachers are encouraged to use activities that promote the joint development and understanding of the geometric and algebraic forms of the parabola.

Finally, it is important that the students develop a facility with both graph sketching and algebraic manipulation. As well, the support that each offers for the other should be emphasized to students. For an example, if the question was to find an equation given a focus and directrix, there are advantages to sketching the graph first.

PROCESS/PROBLEM-SOLVING CONTEXT

A suggestion to start the concept would be to establish conditions for the drawing of a locus that satisfies the conditions given in the definition of a parabola without revealing what shape the result will yield. This can be done either freehand or using paper such as the "circle-line" graphs (see Appendix A). Allow students to make any conclusions that become evident. Once the name of the shape has been determined initiate a discussion to derive the standard forms. Student generated ideas and conclusions can be quite complex as is illustrated by a note in the Mathematics Teacher, Volume 83, Number 2, February 1990, p. 84.

Problem: As a new twist to the problems about suspension bridges, arches and trajectories ask students to speculate about the advantages of knowing the equation and to design questions that can be answered knowing the equation of a particular parabolic structure or phenomenon.

Problem: A spotlight, the back of which is formed by revolving a parabola about its axis of symmetry, has its light source located 20 cm from its vertex. If the light source is at the focus, find the equation of the parabola and discuss what assumptions were made in the solution.

TECHNOLOGY INTEGRATION

As has been mentioned before, graphing calculators and computers now provide students with the opportunity to quickly draw graphs, hypothesize, draw new graphs to check hypotheses and to make conclusions. They aren't perfect, however. Some of the machines may require that the equations be written in the form "y =." Therefore, in graphing an equation of the form $x = y^2$, it will be necessary to graph $y = \sqrt{x}$ and then, using the screen-saving feature, graph $y = -\sqrt{x}$. It may not appear perfectly on the screen (at least in comparison to $y = x^2$) but it is an improvement over having the students use graph paper.

It is probably wise to suggest to students who are considering the purchase of a graphing calculator or some computer software that the form in which the equation has to be given should be one of the purchase criteria.

EXTENSION

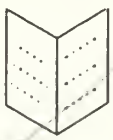
Problem: Consider a problem like this one as a way to integrate a variety of concepts and to use the technology available.

Use a calculator or computer to draw a graph of a parabola. Through experimentation draw a line tangent to the parabola. Now create a line simulating a ray of light that contacts the parabola at the point of contact with the tangent line. Find the equation of the line that would rebound from the parabola through the focus. What about the angles of incidence and reflection?

For example, consider the graph of a parabola of equation $y = x^2$ and a ray of light along the line of equation $x = 1$. The tangent line can be found using calculus and at $x = 1$ its equation would be $2x - y - 1 = 0$. The students should verify that the reflected light will pass through the focus. It's an interesting problem as it combines concepts from two branches of mathematics and also from another discipline, physics. Students with some experience in either or both will probably find the equation suitably interesting.

RESOURCE CORRELATION

HM12:	pp. 160 -164
MPP12:	pp. 311–319
M12:	suppl. pp. 6–13



EXPONENTIAL AND LOGARITHMIC FUNCTIONS
SUGGESTED PROGRAM EMPHASIS – 10%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that an exponential function is one in which the variable appears in the exponent.	Students will be expected to sketch the graph of simple exponential functions of the form $y = a^x$, $a > 0$. Students will be expected to use the graphs of exponential functions to estimate the values of roots and powers.	Students will be expected to draw and analyze the graphs of exponential functions using calculators or computers. Students will be expected to determine the domain and range of the exponential functions. Students will be expected to solve and verify exponential equations.
Students will be expected to demonstrate an understanding that many real world phenomena exhibit exponential properties.	Students will be expected to write exponential functions describing situations involving exponential growth and decay.	Students will be expected to solve problems involving exponential growth and decay.
Students will be expected to demonstrate an understanding of the characteristics and applications of logarithmic functions.	Students will be expected to draw the graphs of logarithmic functions as the inverses of exponential functions. Students will be expected to use the graphs of logarithmic functions to find the values of one of the variables, given the other variable. Students will be expected to convert functions from exponential form to logarithmic form and vice versa.	Students will be expected to draw and analyze the graphs of logarithmic functions using calculators or computers. Students will be expected to determine the domain and range of the logarithmic functions.
Students will be expected to demonstrate an understanding that operations with logarithms are subject to basic properties and laws.	Students will be expected to apply the following laws and properties of logarithms: $\log_a mn = \log_a m + \log_a n$ $\log_a \frac{m}{n} = \log_a m - \log_a n$ $\log_a mn = n\log_a m$ Students will be expected to solve and verify logarithmic equations.	Students will be expected to evaluate logarithmic expressions using calculators and computers. Students will be expected to solve and verify logarithmic equations using calculators or computers.
Students will be expected to demonstrate an understanding that a logarithm with a base of 10 is a common logarithm.	Students will be expected to solve logarithmic equations and evaluate logarithmic expressions using common logarithms.	
Students will be expected to demonstrate an understanding that many phenomena exhibit characteristics that can be described through the use of logarithmic functions.	Students will be expected to write logarithmic functions to describe situations that have logarithmic characteristics.	Students will be expected to solve problems that exhibit logarithmic properties by developing and resolving logarithmic equations.

CONCEPT 1

Students will be expected to demonstrate an understanding that an exponential function is one in which the variable appears in the exponent.

Students will be expected to:

1.1 sketch the graph of simple exponential functions of the form $y = a^x$, $a > 0$

1.2 use the graphs of exponential functions to estimate the values of roots and powers

- 1.2.1 draw and analyze the graphs of exponential functions using calculators or computers
- 1.2.2 determine the domain and range of the exponential functions

1.3 solve and verify exponential equations.

COMMENTS

One way to introduce this concept is to consider an example like the number of direct ancestors one has in one's family. Assume that the first entry is two (representing the student's parents), the second entry is four (for the grandparents), etc. The relation is exponential. It allows the opportunity to investigate domain, range, intercepts and the solution of exponential equations.

Students should have had a good deal of experience graphing various functions but may need to review the graphing of exponential functions.

For this concept the solution of exponential equations should be limited to those that can be solved by transforming both sides to the same base. The solution of other equations should be investigated graphically as indicated in the activity below. Algebraic solutions of these equations will be studied in a later concept in this unit.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Using calculators or computers, have the students draw the graphs of the following: $y = 2x$, $y = x^2$, and $y = 2^x$. Next have the students make a note of any similarities and differences focusing on concepts like intercepts, domain and range. Using information from units studied previously, such as quadratic functions and polynomial functions, have the students predict the differences and the resulting graphs for the following: $y = 4x$, $y = x^4$ and $y = 4^x$. Following this, use the calculator or computer to draw the graphs.

Activity: Using the "zoom" function (or something equivalent) on graphing calculators and computers, determine the values of some expressions and solve some exponential equations using graphs of exponential equations (interpolation). For example,

evaluate: $\sqrt{5}$

solution: rewrite $\sqrt{5}$ as $5^{1/2}$ now draw the graph of $y = 5^x$ "zoom" in on the "y" coordinate of the point that has an "x" coordinate of 1/2 or 0.5

Solve: $20 = 2^x$

Solution: Draw a horizontal line, $y = 20$, intersecting the graph of $y = 2^x$. Then draw a vertical line from this point of intersection to the x-axis to determine that the value of x is approximately 4.3. Therefore $20 \doteq 2^{4.3}$.

If adequate technology is not available, interpolation as discussed in Holtmath 12, p. 128 is an acceptable alternative. This method provides an alternative to approximate the calculation, and reinforces the interpolation studied earlier in mathematics and other disciplines like science and social studies.

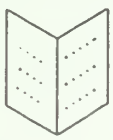
TECHNOLOGY INTEGRATION

The graphing capability of calculators and computers can be very useful tools to assist the student in his process of inquiry. By having the machine do the "leg work" of drawing the graph, the student is then free to do the analysis and hypothesis formulation.

EXTENSION

Let the students experiment with $y = a^{-x}$. This will be useful in later work where exponential growth and decay are studied.

RESOURCE CORRELATION



HM12:	pp. 122–128
MPP12:	pp. 89–94.
AWM12:	pp. 262–265; 274–278

CONCEPT 2

Students will be expected to demonstrate an understanding that many real-world phenomena exhibit exponential properties.

Students will be expected to:

2.1 write exponential functions describing situations involving exponential growth and decay

2.1.1 solve problems involving exponential growth and decay.

COMMENTS

An excellent way to introduce the exponential function is to give students situations where they are required to find a functional relationship between two variables by looking for a pattern. (See PROCESS section below.) Problems that naturally lend themselves to this treatment are those involving percentage growth or decline. Examples are depreciation, population growth, salary increases, reduction in light intensity through various media and compound interest. (Use function notation whenever possible.) Some of these problems will also be dealt with in Geometric Sequences. It is appropriate to deal with problems from different viewpoints. Try to avoid the scenario that results in the students believing there are "five types of exponential problems."

Without other intervening variables, population growth is an exponential function. One approach is to assume that the functional relationship is of the form $P(t) = a \times 10^{k\,t}$ where "a" and "k" are constants that must be determined for the particular situation. (It should be noted that any problem involving exponential growth or decay could be solved this way. Also, the base could be something other than 10.) If one knows the value of P(t) for two values of t, it is possible to solve for the two constants, but because the students have not encountered logarithms yet the solution will be of the "guess and check" variety. (See the PROCESS section below.)

An excellent example of this is the treatment of the rabbit problem in Australia.[1]

The problems involving exponential growth and decay should be limited to those that can be solved algebraically without the use of logarithms. Others may be investigated graphically as in the following activity.

1. Forrester, Paul, Algebra and Trigonometry Functions and Applications, Addison Wesley, 1984, p. 165.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: The value of a house increases by 7% each year. If the initial value is $50,000, produce a table of values for the first 10 years. Look for a pattern in the table and attempt to write a relationship between the value and the time. Write this using functional notation. Use this to find the value of the house after 20 years. Using a graphing calculator, find when the value will be $120,000. (HINT: graph $y=120\ 000$ and the function and look for the point of intersection.) Comment in writing how the value of the house changes through time.

Activity 2 a. Collect information about the population of a country for the last 100 years.

b. Choose an arbitrary starting date and any other date. Assume the population grows exponentially and that the functional relationship takes the form $P(t) = a \times 10^{kt}$ where "a" and "k" are constants that need to be determined.

c. Use this formula to predict the most recent population statistic that you have available. Comment on the accuracy of the calculation.

d. Assume that the population is actually better modelled by using a function of the form $p(t)=ct^2+d$, a quadratic function. Repeat the steps above for this new function.

e. Predict the population of this country in the year 2000 using both functions.

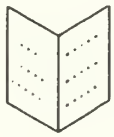
The point of this exercise is that the choice of a function is fairly arbitrary and that a choice should be judged by its ability to make predictions.

Example: The population of the world in 1960 was 3 049 million. The population in 1965 was 3 358 million. Assume that the population can be modelled by a function of the form $P(t)=a\times10^{kt}$.

Choose 1960 as $t=0$. This means that "a" $=3\ 049$. Therefore, 1965 represents $t=5$. By substitution $3\ 358=3\ 049\times10^{5k}$. Dividing both sides leaves $1.103=10^{5k}$. This can be solved by trial and error or by graphing on a graphing calculator and looking for the point of intersection of $y=10^{5x}$ and $y=1.103$. The value of k is approximately 0.0085. Using the function to predict the population in 1988 gives:
$p(28) = 3\ 049\times10^{0.0085(28)} = 5\ 274$ because 1988 is 28 years after 1960, which is our arbitrary $t=0$. The actual answer is 5 143 million.

Problem: The size of a bacteria sample with an initial size of 50, doubles every two hours. Find the size of the sample after eight hours. Use at least two different methods to solve the problem.

RESOURCE CORRELATION



HM12:	pp. 125–128
Nelson:	pp. 96–103
M12:	pp. 262–265; 309–319

CONCEPT 3

Students will be expected to demonstrate an understanding of the characteristics and applications of logarithmic functions.

Students will be expected to:

3.1 draw the graphs of logarithmic functions as the inverse of exponential functions

3.2 use the graphs of logarithmic functions to find the values of one of the variables, given the other variable

- 3.2.1 draw and analyze the graphs of logarithmic functions using calculators or computers
- 3.2.2 determine the domain and range of the logarithmic functions

3.3 convert functions from exponential form to logarithmic form and vice versa.

COMMENTS

1. All the properties of the logarithmic functions can be derived since the logarithmic functions are the inverse of the corresponding exponential functions. The study of logarithmic functions can be an excellent opportunity for students to explore their understanding of functions and their inverses. The properties of functions and their inverses that are required are listed below. Some of these properties were covered in Mathematics 20 but may require review.

   a. A function and its inverse have tables of values that are "mirror" images of one another.
   Therefore, their domains and ranges are interchanged.
   b. If $f(d) = p$ then $f^{-1}(p) = d$.

   The students could graph any logarithmic function and deal with its properties such as domain, range and intercepts without understanding what the function does. It should be pointed out to students that the logarithmic function has a domain where $x > 0$.

2. The students have encountered inverse functions when they solved equations of the following types:

   a. $\sin(x) = 0.5$
   By applying the inverse function to both sides the left side becomes just x.
   $\sin^{-1}(\sin(x)) = \sin^{-1}(0.5)$
   $x = 30°$
   b. $f(x) = x^2 = 45$
   The inverse function of f(x) is the square root function. By applying this to both sides the left side becomes x and the solution is the two square roots of 45.

3. Once the properties of the inverse are understood the student will be able to derive many of the properties of the logarithmic function.
   e.g., if $g(x) = 3^x$ then $g^{-1}(x) = \log_3 x$.

4. Any statement that involves an exponential function can be converted into an equivalent logarithmic statement. The connection can be made through a table of values as follows.

The logic follows these steps:

a. $y = b^x$
b. if (m,n) is on the graph then $n = b^m$
c. this means that (n,m) is on the graph of the inverse
d. the inverse is $y = \log_b x$
e. if (n,m) is on the graph then $m = \log_b n$.

Which means that $n = b^m$ and $m = \log_b n$ are different versions of the same idea. This is an important realization. When a logarithmic equation is given, encourage students to also look at it as an exponential equation.

The functions that were developed in the exponential functions section could be converted to equally valid logarithmic statements. The resulting equation when graphed should be compared to the graph of the corresponding exponential function. Their domains and ranges should also be compared.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Draw a graph of the inverse function of $y = 2^x$.

Problem: What is the domain and range of the function $y = \log_4 x$?

Problem: Without a calculator estimate the value of $\log_2(70)$ to the nearest whole number.

Activity: Describe how you could use the graph of $y = 4^x$ to find the following $\log_4(20)$ and $\log_4(1000)$.

Activity: Draw the graph of $y = 0.5^x$ and its inverse function. What is the inverse function?

TECHNOLOGY INTEGRATION

The graphing calculator or computer will handle graphing the logarithmic and exponential functions. The domain and range of these functions makes it worthwhile to learn how to set ranges for the calculators. This makes more efficient use of the display screen.

EXTENSION

The composition of a function and its inverse is the identity function. This means in this situation that:

a. $g(g^{-1}(v)) = v$ or $3^{\log(v)} = v$
b. $g^{-1}(g(m)) = m$ or $\log_3(3^m) = m$.

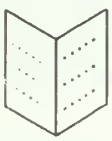
This rule needs to be generalized to other bases.

Activity: Find the inverse of the function $f(x) = 3x + 2$. Demonstrate that the two functions are inverse by choosing several values of x. Show that $f(f^{-1}(a)) = a$ and that $f^{-1}(f(v)) = v$.

Problem: The function $y = \ln(x)$ is a logarithm using an irrational base. By comparing the graph of $\ln(x)$ with that of logarithms with integral bases determine what two integers the irrational base lies between.

Problem: A function called trunc(x) removes all the decimal places of any function it acts on. Comment on the inverse of trunc(x).
e.g., trunc (4.5689) = 4.

RESOURCE CORRELATION



HM12:	pp. 129–132
Nelson:	pp. 110–114
M12:	pp. 279–281; 292-303

CONCEPT 4

Students will be expected to demonstrate an understanding that operations with logarithms are subject to basic properties and laws.

Students will be expected to:

4.1 apply the following laws and properties of logarithms:

$\log_a mn = \log_a m + \log_a n$

$\log_a \frac{m}{n} = \log_a m - \log_a n$

$\log_a m^n = n\log_a m$

4.1.1 evaluate logarithmic expressions using calculators and computers

4.2 solve and verify logarithmic equations

4.2.1 solve and verify logarithmic equations using calculators or computers.

COMMENTS

Some historical background about the development of logarithms by John Napier can set the stage for the study of this concept. Astronomers and other scientists showed great interest in this "new" concept in the early 1600s. They saw its potential for simplifying their computations. This is also a golden opportunity to discuss the developmental nature of mathematics. It isn't often that a topic in secondary mathematics has an origin as directly traceable as does logarithms. Students can explore the "man-made" nature of mathematics by using logarithms as an example. The discussion can be expanded by comparing the computational emphasis from the past with some of the current applications such as logarithmic scales, solution of logarithmic and exponential equations.

Another historical point that students may find interesting is the relationship between the laws of exponents and the laws of logarithms. Most students will readily see the connections and similarities. However, when the laws of logarithms were being developed, exponents were not in common use in Europe. In other words, the two ideas evolved independently and yet are closely related.

As the logarithmic statement is often confusing to students, it is a good idea to emphasize the connection between the logarithmic and exponential functions. Because the latter is more familiar and easier to work with, understanding the laws and properties of logarithms through the understanding of the exponential function should be stressed.

Through emphasis of the connection between the logarithmic and the exponential functions, the students should be able to do verifications similar to the following:

$\log_2(8\times32) = \log_2 8 + \log_2 32$

$\log_2 256 \mid 3 + 5$

$8 \mid 8$

Students will be curious about the "log" button on their calculator so the teacher has a ready-made opportunity to answer the question with a history lesson and have students verify the laws of logarithms using their calculators. There is also the opportunity to compare the values obtained using the exponent functions of the calculator and the logarithmic functions.

It is important to stress that historically logarithms were a great calculation tool but with the advent of calculators, this role is now a minor one. However, more theoretical, algebraic uses have surfaced. The possibilities for equation solving using the laws of logarithms and techniques they have studied previously should be emphasized.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Using their knowledge of exponents and the fact that $\log ab = \log a + \log b$, students could explore cases such as the following:

$\log 5^2 = \log (5 \times 5) = \log 5 + \log 5 = 2 \log 5$
$\log 5^3 = \log (5^2 \times 5) = \log 5^2 + \log 5 = 2 \log 5 + \log 5 = 3 \log 5$
$\log 5^4 \log (5^3 \times 5) = \log 5^3 + \log 5 = 3 \log 5 + \log 5 = 4 \log 5$

Students could be asked to examine the emerging pattern and generalize to $\log 5^n = n \log 5$ for each non-negative integer "n". Students could then use the y^x and log keys on a calculator to confirm this generalization with several numerical values such as $n = 0, 9, 14$.

As in the setting described above, algebraic processing itself often suggests generalizations. Some students might observe, for example, that their algebraic manipulations did not depend on the particular number 5, thus leading to the more general assertion, $\log a^n = n \log a$ for any $a > 0$.

Activity 2: A different and more developmental use of the "log" function of the calculator would be to have the students complete a chart like the one below and then work toward developing some of the concepts themselves.

m	n	logm	logn	mn	log(mn)	10(m+n)	10 log(mn)

Problem: Given the following approximations $\log_2 3 = 1.585$, $\log_2 5 = 2.322$ and $\log_2 10 = 3.322$, approximate the base 2 logarithm of the following numbers:

a. 1.5 b. 20

Solution: a. $1.5 = \frac{3\times 5}{10}$, $\log_2\left(\frac{3\times 5}{10}\right) = \log_2 3 + \log_2 5 - \log_2 10$

$= 1.585 + 2.322 - 3.322$

$= 0.585$

b. $20 = 4\times 5$, $\log_2(4\times 5) = \log_2 4 + \log_2 5$

$= 2 + 2.322$

$= 4.322$

Problem: Solve: $\log_2 x + \log_2(x+2) - 3 = 0$

Solution: $\log_2(x^2+2x) = 3$

$x^2 + 2x - 8 = 0$

Therefore $x = -4$ and $x = 2$.

One of these won't check. Which one? Why?

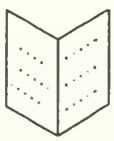
TECHNOLOGY INTEGRATION

The calculator has replaced logarithms as a basic computational tool but there are numerous opportunities for logarithms and calculators to be used together to solve new problems. Some of these new applications have been illustrated in the above section.

EXTENSION

The teacher is referred to an article in the Mathematics Teacher, Volume 78, Number 2, February 1985, pp. 106–109 for an interesting approach to proving the laws of logarithms.

RESOURCE CORRELATION



HM12: pp. 133–141
MPP12: pp. 115–116; 119–128
M12: pp. 282–291; 304–308

CONCEPT 5

Students will be expected to demonstrate an understanding that a logarithm with a base of 10 is a common logarithm.

Students will be expected to:

5.1 solve logarithmic equations and evaluate logarithmic expressions using common logarithms.

COMMENTS

Students should be aware that log x means $\log_{10}x$ and that the log button on the calculator gives the common logarithm (which is base 10).

Students should be expected to solve exponential equations of the form $y=a^x$ for "x" or for "a" by using the fact that if $y=a^x$ then $\log y=\log a^x$. Examples of these are $2=3^x$ and $x^{2/3}=5$.

Students will be expected to find the logarithm of any positive number to any positive base using common logarithms. For example, evaluate $\log_3 6$. This can be done by letting $\log_3 6=x$, changing this to its equivalent exponential form and then solving this equation using common logarithms.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem: The number 64 can be expressed as a power of 2, 4 and 8 using integral exponents. Can you write 64 as a power of 10? Can you write 64 as a power of 5?

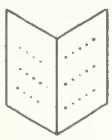
TECHNOLOGY INTEGRATION

The logarithmic and exponential functions are inverse functions. On the calculator, students could use the y^x and log keys to confirm this generalization with numerical values.

EXTENSION

Students could be shown the tables of common logarithms and learn how common logarithms were found before we had calculators.

RESOURCE CORRELATION



HM12:	pp. 137–139
MPP12:	pp. 130–131
M12:	pp. 279–281

CONCEPT 6

Students will be expected to demonstrate an understanding that many phenomena exhibit characteristics that can be described through the use of logarithmic functions.

Students will be expected to:

6.1 write logarithmic functions to describe situations that have logarithmic characteristics

6.1.1 solve problems that exhibit logarithmic properties by developing and resolving logarithmic equations.

COMMENTS

This concept provides an excellent opportunity to integrate topics from areas such as chemistry and physics. Magnitudes of earthquakes, loudness of sound, intensity of light pH are all measured using logarithms. Students should be familiar with these examples and have an understanding of why they are used.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Have students figure out their ages and the ages of the other members of their family in seconds and write the ages in a chart from smallest to largest. Have them set up a logarithmic scale to represent these ages and convert each of the ages in seconds to their new scale. Once this scale is developed ask the students such questions as "Does a number twice as big on the new scale make you twice as old?" "What will the reading on the new scale be if you are twice as old?" "Will you be alive when your scale reading is one more than it is now?"

Sample Chart

Age in years	Age in seconds	Age on log scale using Age = log (age in seconds)
12	378432000	8.58
17	536112000	8.73
40	1261440000	9.10
42	1324512000	9.12

Why is it more convenient to deal with very large or very small numbers using logarithms?

Activity: Have students investigate why Richter developed the Richter scale to measure the magnitude of earthquakes, and note that it uses logarithms.

Problem: The half-life of strontium 90 is 25 a. How long does it take for a sample of 150 mg to decay to a mass of 60 mg?

Solution: Using $M(t) = m_0 \times (\frac{1}{2})^{t/h}$ or a formula similar to this

$60 = 150 \times (\frac{1}{2})^{t/25}$

$0.4 = (\frac{1}{2})^{t/25}$

By taking the logarithm of each side of the equation, $t = 33$a.

Problem: The Alaska earthquake of 1964, which had a magnitude of 8.5 on the Richter scale, was 40 times more intense than the San Francisco earthquake of 1989. What did the San Francisco earthquake measure on the Richter scale?

Solution: $10^{8.5-R} = 40$. By taking the common logarithm of both sides

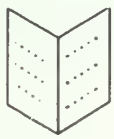
$8.5 - R = 1.6$

$6.9 = R$

TECHNOLOGY INTEGRATION

The log function on the calculator and its inverse function should be used.

RESOURCE CORRELATION



HM12:	pp. 142–144
MPP12:	pp. 133–137
M12:	pp. 309–319

PERMUTATIONS AND COMBINATIONS
SUGGESTED PROGRAM EMPHASIS – 10%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding of the Fundamental Counting Principle.	Students will be expected to calculate the total number of ways that a multiple of tasks can be conducted if each task can be performed in a multiple of ways.	Students will be expected to solve problems that involve the use of the fundamental counting principle.
Students will be expected to demonstrate an understanding that a permutation is an arrangement in which the order is important.	Students will be expected to calculate the number of permutations there are of n things taken k at a time by applying the formula ${}_nP_k = n!/(n-k)!$	Students will be expected to solve problems involving linear permutations, permutations with repetitions, circular permutations and permutations with reflections. Students will be expected to solve probability questions that involve the use of permutations.
Students will be expected to demonstrate an understanding that a combination is an arrangement in which the order is not important.	Students will be expected to calculate the number of combinations there are of n things taken k at a time by applying the formula ${}_nC_k = n!/k!(n-k)!$	Students will be expected to solve probability problems that involve the use of combinations.
Students will be expected to demonstrate an understanding that the numerical coefficients of the terms in a binomial expansion can be determined by using the Binomial Theorem.	Students will be expected to expand binomials of the form $(x+a)^n$, $n \in W$ using the Binomial Theorem. Students will be expected to relate the numerical coefficients in a binomial expansion to the terms of Pascal's Triangle and vice versa.	

CONCEPT 1

Students will be expected to demonstrate an understanding of the Fundamental Counting Principle.

Students will be expected to:

1.1 calculate the total number of ways that a multiple of tasks can be conducted if each task can be performed in a multiple of ways

1.1.1 solve problems that involve the use of the fundamental counting principle.

COMMENTS

Students from Mathematics 20 and Mathematics 33 have studied probability and so have been exposed to counting the number of outcomes possible for certain events, such as rolling two dice or flipping three coins, using tree diagrams. Students from Mathematics 33 studied probability in Mathematics 23.

This concept should be approached with a problem-solving focus where students investigate solutions that involve counting theory. Use of familiar experiences requiring an arrangement of letters or digits such as licence plate numbers, telephone numbers, social insurance numbers, or combinations for locks on their lockers can be used.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have students choose any four digits and count how many different numbers they can form using these four digits (a) without repetition and (b) with repetition. Compare their results with other students. Have them investigate a possible way of counting the number of different arrangements without having to list them.

Problem 1: Many licence plates in Canada and the United States consist of three letters followed by three numbers. How many licence plate numbers can be formed using three letters followed by three digits?

Activity 2: Personalized licence plates are now available. Have students investigate the restrictions placed on these licence plates and then determine the total number of personalized licence plates possible. See the Mathematics Teacher, Volume 77, Number 3, March 1984, p. 183, for an article titled "Counting Using Licence Plates and Phone Numbers: A Familiar Experience."

Problem 2: How many radio station call letters are available in Canada if they must consist of three or four letters and the first letter must be C?

Solution:

Using four letters	$1\times26\times26\times26$	$= 17576$
Using three letters	$1\times26\times26$	$= 676$
Total		18252

An alternative solution addresses the three-letter case by assigning a blank as a possible fourth letter.

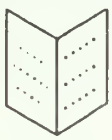
$1\times26\times26\times27=18252$

See questions in MPP30 p. 405 and in AW30 Number 17 and Number 19, p. 502.

EXTENSION

For other methods of counting, see Niven, Ivan, Mathematics of Choice, How to Count Without Counting. The Mathematical Association of America. This book is an excellent resource for questions on permutations and combinations.

RESOURCE CORRELATION



AW30:	pp. 500–502
HM12:	pp. 349–352
MPP30:	pp. 402–405

CONCEPT 2

Students will be expected to demonstrate an understanding that a permutation is an arrangement in which the order is important.

Students will be expected to:

2.1 calculate the number of permutations there are of n things taken k at a time by applying the formula ${}_nP_k = n!/(n-k)!$

- 2.1.1 solve problems involving linear permutations, permutations with repetitions, circular permutations and permutations with reflections
- 2.1.2 solve probability questions that involve the use of permutations.

COMMENTS

Factorial notation must be discussed and should be evaluated on a calculator.

Emphasize that in an arrangement of objects, a change in the order represents a different permutation.

The development of the formula for ${}_nP_k$ should be discussed. Students must first discover that the number of ways of arranging n of n objects is n! and recognize that ${}_nP_n$ is the symbol used to represent the number of ways of arranging n of n objects. To determine the number of permutations of n objects taken k at a time where k is less than n, use the fundamental counting principle to establish that ${}_nP_k = n(n-1)(n-2)\ldots(n-k+1)$.

And since $n! = n(n-1)(n-2)\ldots(n-k+1)(n-k)!$

$n! = {}_nP_k(n-k)!$

Thus $n!/(n-k)! = {}_nP_k$.

To investigate why 0! is defined as 1, let $k = n$ in the formula above.

Thus ${}_nP_n = n!/(n-n)!$

${}_nP_n = n!/(0)!$ and since ${}_nP_n = n!$, 0! must be equal to 1.

The formula for ${}_nP_k$ can be used only when no restrictions or duplications occur in the problem. In these situations the fundamental counting principle must be used.

Permutations involving special situations such as adjacent objects, permutations with repetitions, circular arrangements, ring permutations and permutations involving reflections should be introduced by having students investigate the number of permutations by counting them and then discovering a formula that can be used for different situations that involve such arrangements. Use four objects to keep the counting reasonable.

Problems involving adjacent objects require that a certain number of the objects be next to one another in all of the arrangements. Treat the objects that are to be adjacent as a unit and determine the number of permutations. This number is multiplied by the number of ways the adjacent objects can be arranged since in each of the original permutations the objects are in the same order. The number of ways that n objects can be arranged if k of the objects must be adjacent is $(n-k+1)!k!$.

Permutations with repetitions have the same element occurring more than once in the arrangement. Treat all of the objects as if they are distinct and find the number of arrangements possible. Then divide this number by the number of ways each of the like objects can be arranged. The number of arrangements of n objects if p objects are the same, q items the same and so on is $n!/p!q!\ldots$.

Circular permutations require that the objects be arranged in a circle rather than a line. In a circular arrangement a rotation of the arrangement is not considered a different permutation. To determine the number of circular permutations place an object on the circle, then arrange the others with respect to this object. The number of ways that n objects can be arranged in a circle is $(n-1)!$ since only the position of the articles relative to one another is important. In some situations the objects can also be reflected without changing the permutation. For example, if a key ring is turned over (that is, reflected) this arrangement may look different but we do not consider the permutation to be different. A permutation that is considered the same if it is rotated and reflected is called a ring permutation. The number of ring permutations of n objects is $(n-1)!/2$.

Probability is an application of this concept, not the focus. Students from Mathematics 20 and Mathematics 33 have studied probability. Review of the terminology used, such as sample space, outcomes, favourable outcomes, events and the formula for finding the probability of an event may be necessary. Care must be taken in selecting questions on probability from the resources. Questions that involve permutations rather than combinations must be selected at this point.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1: In how many zeros does 100! end? Answer: 24.

Problem 2: On a used car lot eight cars to be sold are to be arranged side by side in the front row facing the street.

a. How many ways can this be done if the three black cars must be together?
b. How many ways can the eight cars be arranged if all three black cars must not be adjacent to one another?

Solution:

a. The three black cars are considered one unit, so there are the five other cars and this unit to be arranged; that is, six items to be arranged. This can be done in 6! or 720 different ways. However, in each of these arrangements there are 3! ways of arranging the black cars. Therefore, there are $6! \times 3! = 720 \times 6 = 4320$ ways of arranging the cars where the black cars are adjacent.

b. Since there are 8! ways of arranging the cars with no restrictions and 4320 of these ways have the three black cars adjacent, the remaining arrangement will not have the three black cars next to one another, so there are $8! - 4320 = 36\ 000$ ways of arranging the cars where the three black cars are not adjacent.

Problem 3: A new computer company will give away a million dollars to anyone who guesses the order into which they have arranged the letters of the word SUPERCOMPUTER. Each entry must be in a separate envelope. How much would it cost you in postage to submit all possible entries? After figuring out this expense, you decide to send in 13 different entries since 13 is your lucky number. What is the probability that you will win the million dollars?

Solution: Temporarily place subscripts on the letters that are alike. To distinguish between the first U and second U use U_1 and U_2 and do the same for the two Ps, two Es, and two Rs. If the 13 letters of the word SUPERCOMPUTER were all different there would be 13! ways of arranging them. Half these arrangements will contain the Us in the order U_1, U_2 and the other half will contain them in the order U_2, U_1. Without the subscripts, these two groups of permutations are indistinguishable. Hence the number of distinguishable permutations is 13!/2! just because of the Us being alike.

Similarly, of the arrangements remaining, many differ only in the arrangements of the Ps, or in the arrangement of the Es and of the Rs. Thus the number of distinguishable permutations is 13!/2!2!2!2!
The cost is 13!/2!2!2!2!2! × cost of one postage stamp.
The probability of winning is 13/(13!/2!2!2!2!).

Problem 4: A necklace is made by stringing wooden beads on a leather thong that is tied at one end.

a. If 12 beads of various sizes are available and all are placed on the thong and a knot is tied at the other end, how many different necklaces could be made?

b. If the 12 beads were placed on a string in a circular arrangement, that is, one where the knot is not visible, how many different necklaces could be made?

Solution:

a. There are 12! ways of placing the beads on the thong, but because the exact opposite order of the beads does not produce a different necklace, there are 12!/2 different necklaces possible.

b. Because only the arrangement of the beads relative to one another is distinguishable, place one bead on the string and determine the number of ways of arranging the remaining beads on the string; thus 11!. Then, because of the reflective property of the necklace, the number of permutations is one-half of the 11!; that is 11!/2.

The article "Exotic Horse-Race Wagering and Combinatorics" in Mathematics Teacher, Volume 77, Number 1, January 1984, p. 35, presents some bets that could be used to introduce permutations.

TECHNOLOGY INTEGRATION

Use of the factorial key, and ${}_nP_r$ key should be shown. Not all calculators have both these keys. Have students investigate the use of these keys by doing an exercise such as AW30 p. 508 INVESTIGATE at the bottom of the page. Care must be taken in using the ! key because even small numbers give answers in scientific notation.

Computers can be programmed to calculate permutations. MPP12 p. 430 presents one such program and exercise.

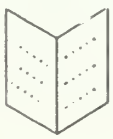
Mathematics Teacher, Volume 77, Number 1, January 1984, has an article on calculating factorials beyond 70!.

Computers can be programmed to print out the possible permutations of a certain number of objects. See Mathematics Teacher, Volume 80, Number 5, May 1987, p. 403, for a possible program.

EXTENSION

Investigate the number of circular or ring permutations if some of the elements are the same. For example, two red beads, and three white beads are used to make a necklace in a circular arrangement. How many different necklaces could be made? Can the student develop a formula for this?

RESOURCE CORRELATION



MPP12:	pp. 407–410
	pp. 418–432
HM12:	pp. 353–357
	pp. 363–371
AW30:	pp. 503–512

CONCEPT 3

Students will be expected to demonstrate an understanding that a combination is an arrangement in which the order is not important.

Students will be expected to:

3.1 calculate the number of combinations there are of n things taken k at a time by applying the following formula:
$_nC_k = n!/k!\,(n-k)!$

3.1.1 solve problems including probability problems that involve the use of combinations.

COMMENTS

Emphasize that a combination is a selection in which order is not important. Encourage students to ask themselves if order is important or not before a question is begun.

The notation for combinations varies in the resources. $_nC_r$ and $\binom{n}{r}$ are equivalent notations. The formula for $_nC_r$, like the formula $_nP_r$, can only be used when objects are distinct and repetitions are not allowed.

The relationship between $_nP_r$ and $_nC_r$ should be discussed. With n distinct objects, $_nC_r$ counts the number of ways of choosing r of them without regard to order. Any one of these choices is simply a collection of r objects. Such a collection can be ordered in r! ways. Thus, there are r! times as many permutations as there are combinations, therefore $_nP_r = r! \times {}_nC_r$ or $_nC_r = {}_nP_r/r! = n!/r!\,(n-r)!$

Probability is not the focus of this concept, merely an application. Students from Mathematics 20 and Mathematics 33 have worked with the multiplication and addition rules for probability.

The problems that follow are challenging applications of this concept.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1: A student lives in a house that is five blocks west and seven blocks south of the school she attends. Thus, walking to school each day she goes 12 blocks. All the streets are in a rectangular pattern and are available to her for walking. In how many different ways can she go to school, walking only 12 blocks?

Solution: Let W denote one block west and S denote one block south. Therefore, one path could be WWSWSSSWWSSS, which means she goes two blocks west, then one block south, then one block west and three blocks south, etc. Thus each path consists of a string of five Ws and seven Ss interspersed in a row. To determine the total number of paths decide the number of ways to fill five of the 12 spaces with Ws and thus the remaining spaces with Ss. This is $_{12}C_5 = 792$. Note this is also the number of ways we can choose seven out of the 12 spaces to fill with Ss.

Problem 2: The Stanley Cup in hockey consists of a sequence of games that terminates when one team (the winner) wins its fourth game. If the two teams are equally likely to win any one game, what is the probability that the series will last exactly four games? Five? Six? Seven? (Assume that outcomes of successive games are independent.)

Solution: Call the two teams A and B. The probability that A will win any one game is 1/2, that is $P(A)=1/2$ and similarly, $P(B)=1/2$. For the series to be won in exactly four games the two sequences of victories are AAAA and BBBB so the probability that the series will be won in exactly four games is $(1/2)^4+(1/2)^4=0.125$. Sequences that end the series in five games are BAAAA, ABAAA, AABAA and AAABA with B and A interchanged. The probability the series lasts exactly five games is $4P(A)^4P(B)+4P(A)P(B)^4=0.25$. The number of ways the series will last six games with A winning the series is the number of ways we can choose three from the first five games for A to win (since for A to win the series A wins the last game and three of the first five games). Similarly, the number of ways B will win the series in six games is ${}_5C_3=10$. Therefore, the probability that the series will last exactly six games is $10P(A)^4P(B)^2+10P(A)^2P(B)^4=0.3125$. Continuing this reasoning the probability the series will last seven games is $20P(A)^4P(B)^3+20P(A)^3P(B)^4=0.3125$.

How would these probabilities change if the probability of team A winning any one game is 0.6? Answers to two decimal places are 0.16, 0.27, 0.30 and 0.28.

TECHNOLOGY INTEGRATION

Use of the ${}_nC_r$ key on the calculator.

Computers can be programmed to calculate the number of combinations. See MPP12, p. 430 for one program.

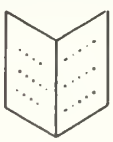
Computers can be programmed to print combinations. See Mathematics Teacher, Volume 80, Number 5, May 1987, p. 403.

EXTENSION

1. The fundamental counting principle allows us to determine the number of outcomes possible where repetition is allowed and order is important. Our study of permutations allows us to determine the number of outcomes possible where repetition is not allowed and order is important. Our study of combinations allow us to determine the number of outcomes possible where repetition is not allowed and order is not important. Is there a way of determining the number of outcomes possible where repetition is allowed and order is not important? For example, in rolling dice we usually determine the number of outcomes possible by distinguishing the dice by making them different colours. So in rolling four dice, there are 6^4 possible outcomes. The outcome (1, 2, 3, 3) is not the same as (1, 3, 2, 3) or (3, 3, 2, 1). However, if the dice were all the same colour, these three outcomes would all look the same. In rolling four dice how many visually distinct outcomes are possible if the dice are identical? (12.6) In Mathematics Teacher, Volume 81, Number 2, February 1988, p. 146 the formula ${}_nW_k = {}_{n-1+k}C_k = (n-1+k)!/(n-1)!k!$, in honour of Bishop Wibold, is developed to represent the number of visually distinct outcomes that can occur when a set of n objects is sampled k times with replacement but without respect to order.

2. The article "Counting Pizza Pieces and Other Combinatory Problems" in Mathematics Teacher, Volume 81, Number 1, January 1988, p. 22 establishes a formula to represent the maximum number of pieces into which a pizza (circle) can be divided by n cuts. It also presents a solution and formula for finding the maximum number of regions into which the interior of the circle can be divided by placing n points on the circle and connecting each pair of points. These formula are developed by counting and analyzing the results.

RESOURCE CORRELATION



MPP12: pp. 411–415
pp. 417–423
HM12: pp. 358–360
pp. 363–364
pp. 368–371
AW30: pp. 513–516

CONCEPT 4

Students will be expected to demonstrate an understanding that the numerical coefficients of the terms in a binomial expansion can be determined by using the Binomial Theorem.

Students will be expected to:

4.1 expand binomials of the form $(x+a)^n$, $n \in W$ using the Binomial Theorem

4.2 relate the numerical coefficients in a binomial expansion to the terms of Pascal's Triangle and vice versa.

COMMENTS

Introduce Pascal's Triangle through evaluating $\binom{n}{r}$ and writing them in the form of a triangular array. Once the first three or four rows are completed have students develop a method of completing the following three or four rows. Pascal's Triangle can be used in determining the coefficients of the terms in binomial expansions so should be discussed before the binomial theorem.

Squaring a binomial should be familiar to students in Mathematics 30. Expanding the cube of a binomial or finding the expansion of higher powers of a binomial may need some explanation. To discover how Pascal's Triangle can be useful have students expand binomials of the form $(a+b)^n$ where $n=0$, 1, 2, 3, etc. and notice that the expansions proceed in decreasing powers of a and increasing powers of b, their exponents always adding to the exponent in the binomial. Their numerical coefficients pattern those in Pascal's Triangle. The numerical coefficient of each term should also be discussed in terms of the number of ways of choosing a letter from each of the n factors $(a+b)$. This is discussed in all texts.

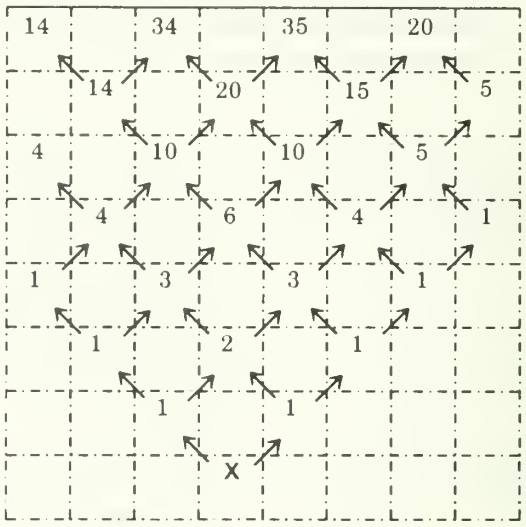
In expanding binomials of the form $(ax+by)^n$ where a and b are constants, the numerical coefficients of each term in the expansion are also determined by the values of a and b. Also, since the sequence of terms in each expansion follows a definite pattern, an expression for the general term t_n (or t_{n+1}) can be developed.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: After studying Pascal's Triangle have students find patterns in Pascal's Triangle. For example, by adding the numbers in each row, the powers of 2 appear. By adding the squares of the numbers in each row a pattern appears that can be seen in Pascal's Triangle. For more patterns involving the counting numbers, triangle numbers and pyramid numbers see "The Twelve Days of Christmas and Pascal's Triangle" in Mathematics Teacher, Volume 75, Number 9, December 1982, p. 755.

Problem 1: An eight-square by eight-square game board has a checker positioned in the fourth square of the bottom row. The checker is allowed to move one square at a time diagonally left or right to the row above. After seven moves the checker will be in the top row. How many different paths will lead to the top row?

Answer:



There are $14+34+35+20=103$ ways of moving to the top row.

The first move can be into one of the indicated positions in the second row. There is one way to move to each of these positions. The next move is to one of the three squares in the third row from the bottom. There is only one way to get to each of the squares on the outside, but two ways to move to the middle position. This reasoning is continued to solve the problem.

Repeat this question when the checker is placed in the bottom right and corner. Answer is 35.

Activity 2: Have students simplify $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$ using their previous skills with polynomials. By looking at these expansions have them find patterns in the number of terms, the degree of each term and the numerical coefficients of the terms. Have them find the expansion of $(a+b)^5$ and $(a+b)^6$ using these patterns.

Problem 2: The polynomial $(p+q)^8$ is expanded in decreasing powers of p. The second and third terms have equal values. If p and q are positive numbers whose sum is 2, find the value of p.

Solution: Set up a system of equations to solve. One equation is $p+q=2$. The other equation is $8p^7q=28p^6q^2$.
The value of p is 8/5.

Problem 3: In the expansion of $(x^2 - 1/x)^9$ find:

a. the general term t_{r+1}
b. the term containing x^6
c. the constant term.

Solution:

a. $t_{r+1} = \binom{9}{r} (x^2)^{9-r}(-1/x)^r$

$= \binom{9}{r} (-1)^r x^{18-3r}$

b. Thus form the term containing x^6, $18 - 3r = 6$ so $r = 4$. Therefore the fifth term is $126x^6$.

c. For the constant term $18 - 3r = 0$, therefore, $r = 6$. Thus the seventh term is the constant term, which is equal to 84.

Some other exercises involving the binomial expansion involve expanding the binomial using the binomial theorem and then multiplying by another binomial.

Example 1: Expand $(x^2 + 2)(x + 1/x)^5$.

Answer: $x^7 + 7x^5 + 20x^3 + 30x + 25/x + 11/x^3 + 2/x^5$

Example 2: Find the first three terms in the expansion of $(x + 1)^5 (x - 2)^6$.

Answer: $x^{11} - 7x^{10} + 10x^9 + \ldots$

TECHNOLOGY INTEGRATION

For a computer program that prints the expansion of binomials see Computer-Application Binomial Expansion, HM12, p. 755.

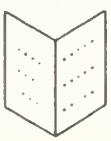
For a program to print Pascal's Triangle see Mathematics Teacher, Volume 76, Number 9, December 1983, p. 686.

EXTENSION

Binomial Probability, MPP12: pp. 426–428.

Multinomial Expansions: Expand $(x + y + z + t)^3$. See Niven, Ivan, Mathematics of Choice or How to Count Without Counting, The Mathematical Association of America.

RESOURCE CORRELATION



HM12: pp. 361–362, 372
MPP12: pp. 424–425
AW30: pp. 517–519

SEQUENCES AND SERIES
SUGGESTED PROGRAM EMPHASIS – 10%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that a sequence is a set of quantities determined by a rule (function) whose domain is the natural numbers and whose range is the terms of the sequence.	Students will be expected to recognize finite and infinite sequences. Students will be expected to write the terms of a sequence given the function that defines it. Students will be expected to write the terms of a sequence given its recursive definition. Students will be expected to determine the function that describes a simple sequence.	
Students will be expected to demonstrate an understanding that a series is the sum of the terms of a sequence.	Students will be expected to expand a series defined using sigma notation.	
Students will be expected to demonstrate an understanding that arithmetic sequences are such that each term is equal to the sum of the preceding term and a constant and that an arithmetic series is the indicated sum of the terms of an arithmetic sequence.	Students will be expected to apply the general term formula of arithmetic sequences, $t_n = a + (n-1)d$. Students will be expected to apply the sum formula for arithmetic series, $S_n = \frac{n(a+t_n)}{2}$, $S_n = \frac{n[2a+(n-1)d]}{2}$	Students will be expected to solve problems involving the use and application of the general term formula for arithmetic sequences. Students will be expected to solve problems involving the use and application of the sum formula for arithmetic series. Students will be expected to use technology where applicable.
Students will be expected to demonstrate an understanding that geometric sequences are such that each term is equal to the product of the preceding term and a constant and that a geometric series is the indicated sum of the terms of a geometric sequence.	Students will be expected to apply the general term formula of geometric sequences, $t_n = ar^{n-1}$. Students will be expected to apply the sum formula for geometric series, $S_n = \frac{a(r^n-1)}{(r-1)}$	Students will be expected to solve problems involving the use and application of the general term formula for geometric sequences. Students will be expected to solve problems involving the use and application of the sum formula for geometric series. Students will be expected to use technology where applicable.

CONCEPT 1

Students will be expected to demonstrate an understanding that a sequence is a set of quantities determined by a rule (function) whose domain is the natural numbers and whose range is the terms of the sequence.

Students will be expected to:

1.1 recognize finite and infinite sequences
1.2 write the terms of a sequence given the function that defines it
1.3 write the terms of a sequence given its recursive definition
1.4 determine the function that describes a simple sequence.

COMMENTS

An effective method to introduce this unit is to use problems that generate sequences. Most students will solve these by trying to solve a simpler problem and looking for a pattern to emerge. The resulting numbers form a sequence. This is a strategy that is commonly used to introduce algebraic concepts in Mathematics 10. The terminology and basic concepts of sequences can be taught through these questions. This is often more effective than introducing a sequence as a theoretical construct.

The next step is to find a functional relationship between the variables involved. The possibilities for these functions include many of the types that students have discussed in Mathematics 20 and Mathematics 30. A key difference, however, is the restriction of the domain to the natural numbers. It is important for students to realize that any function previously studied can form a sequence. Graphing a sequence is an excellent method of emphasizing the connections and contrasts to previous concepts.

The notation used in sequences can be quite confusing. It is useful to compare and contrast function and sequence notation.

Example:
$f(3) = t_3$ means that the third term is obtained by evaluating the general term with $n = 3$. (See Activity 1.)

Many of the sequences generated will have polynomial functions as the defining rules. Students should be given opportunities to find the rules through various means including guess and check, establishing a recognizable pattern and perhaps a specific technique such as finite differences. An excellent article has been included in this document showing how to develop and apply this technique.[1] Knowledge of this technique is **not** a mandatory component of the curriculum but may be an effective teaching tool.

1. Guillote, Henry. "The method of finite differences: Some applications," Mathematics Teacher, Volume 79, Number 6, p. 466.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1:

a. Give an example of any polynomial, logarithmic, exponential and trigonometric function.
b. List the first five terms of the sequence that would result. Use both function and sequence notation to list the terms.
c. Graph each set of five points.
d. List the domain and range of each function below the graph.

Example:
$q(n) = 2^n$ is an exponential function
$q(1) = t_1 = 2$; $q(2) = t_2 = 4$; $q(3) = t_3 = 8$ etc.

Problem 1: Thirty delegates arrive at a conference on the environment. If each delegate shakes hands with every other delegate, how many handshakes will there be in total?

Solution: Students will probably be able to predict the total number of handshakes by developing a pattern similar to the one below. The problem that should then be raised is, can a formula be written that will will give the number of handshakes as a function of the number of people in the room. Use function notation whenever possible. (Example: $h(4) = 6$ means that with four people there are six handshakes.) The solution outlined in the table below uses Finite Differences but other methods are possible. For example, if the students visualize n people in a room then there are $(n-1)$ people to shake hands with. This would give a total of $n(n-1)$ handshakes. However, each handshake would have been made twice so the real total is half of that or $\frac{n(n-1)}{2}$.

Number of People (n)	Number of Handshakes	Diff. #1	Diff. #2
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		1
		5	
6	15		

Using the results of the method of finite differences the relationship must be second degree of the form $an^2 + bn + c$.

Also using finite differences gives:

$a = \frac{1}{2}, b = -\frac{1}{2}, c = 0.$

Therefore: $t_n = h(n) = \frac{1}{2}n^2 - \frac{1}{2}n$ or $t_n = \frac{n(n-1)}{2}$

and $t_{30} = h(30) = 435$ handshakes.

Here is an additional problem that students may have encountered in Mathematics 10. A functional relationship may be difficult to find without a specific technique like Finite Differences. Assignments should be limited to determining functions describing simple sequences. These could include quadratic or cubic functions with angle terms, or arithmetic and geometric sequences.

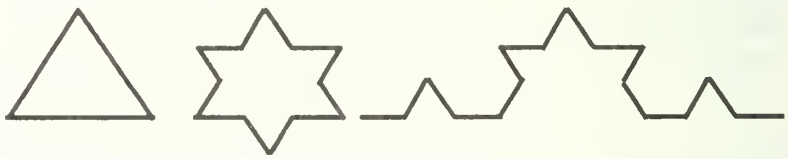
Problem 2: How many squares can be drawn on an 8×8 checkerboard?

Size of Board (n)	Squares (s)	Diff. #1	Diff. #2	Diff. #3
1×1	1			
		4		
2×2	5		5	
		9		2
3×3	14		7	
		16		2
4×4	30		9	
		25		2
5×5	55		11	
		36		
6×6	91			

The relationship is of the form $s(n) = an^3 + bn^2 + cn + d$. Solving for the coefficients gives $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}$ and $d = 0$.

Therefore: $t_8 = s(8) = 204$ squares.

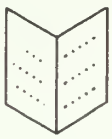
Activity: An interesting activity that could be used as an introduction or extension is the exploration of the "snowflake curve." This is a shape that is formed by subdividing the sides of an equilateral triangle. In each stage of construction every side is replaced by four new sides with length one-third of the side that was replaced. Many sequences can be conceived from the resulting figures. The number of sides, the length of each side or the total perimeter can be investigated. This is explored in some detail in an article by John Egsgard.[1]



(See Mathematics 23 TRM, p. 177 for a program that generates the curve.)

1. Egsgard, John.

RESOURCE CORRELATION



AW:	pp. 328–331
Holt:	pp. 418–420
MPP12:	pp. 349–353

CONCEPT 2

Students will be expected to demonstrate an understanding that a series is the sum of the terms of a sequence.

Students will be expected to:

2.1 expand a series defined using sigma notation.

COMMENTS

A series is formed by summing the terms of a sequence. This can be introduced through problems whose solution requires terms to be added together. It is important to emphasize the connection between series and sequences. The use of sigma notation is an excellent bridge.

Example:

$\sum_{n=1}^{5} (2n+1)$ Here $(2n+1)$ represents the general term of the sequence. The series is the sum of the first five terms of this sequence.

The sum, S, that is obtained is a function of the number of terms, n, that are added (i.e., S(n) if we use function notation or S_n using series notation).

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1: If the song "The Twelve Days of Christmas" were extended to 30 days then how many gifts would be given by the end of the thirty days. On the first day one gift is given, on the second two are given until 30 would be given on the last day. Some students will not see the difference between finding the number of gifts given on the thirtieth day and the total given.

Solution: $\sum_{n=1}^{30} (n)$ would be the sigma notation. Students could then determine the sum of the terms. Also see the EXTENSION section.

EXTENSION

Problem 1 in this concept could then be solved in the following manner.

Solution: We are looking for S(30) if we think of "S", the total number of gifts, as a function of "n," the number of days.

Day (n)	No. of Gifts Given	Total Gifts Given (S)	Diff. #1	Diff. #2
1	1	1		
			2	
2	2	3		1
			3	
3	3	6		1
			4	
4	4	10		1
			5	
5	5	15		

The general term of the sequence of partial sums (the third column) is a second degree polynomial. (Using finite differences.) The general form of this term is:
$S(n) = an^2 + bn + c$.

Using the results of the method of finite differences gives the following results:
$a = \frac{1}{2}$; $b = \frac{1}{2}$; $c = 0$.

Therefore using functional notation:
$S(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

Using the series notation would give:
$S_n = \frac{1}{2}n^2 + \frac{1}{2}n$, where S_n represents the sum of n terms of a sequence.

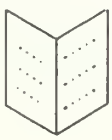
Another way to state this result is that:
$1 + 2 + 3 + 4 + 5 \ldots + n = \frac{1}{2}n^2 + \frac{1}{2}n$.

The answer to the question is then $S_{30} = S(30) = 465$ gifts.

Students could be asked to find the formula for the sum of a series. If the formula can be written as a polynomial then the method of finite differences can be used. This is accomplished by forming the sequence of partial sums. Refer to the article contained in this document for details.[1] Other methods of finding the general rule, such as guess and check, are acceptable methods of solution.

1 Guillote, Henry P. "The Method of Finite Differences: Some Applications," Mathematics Teacher, Volume 79, Number 6, September 1986, p. 466.

RESOURCE CORRELATION



AW:	pp.
Holt:	pp. 432, 433
MPP12:	pp. 364–366

CONCEPT 3

Students will be expected to demonstrate an understanding that arithmetic sequences are such that each term is equal to the sum of the preceding term and a constant and that an arithmetic series is the indicated sum of the terms of an arithmetic sequence.

Students will be expected to:

3.1 apply the general term formula of arithmetic sequences, $t_n = a + (n-1)d$.
- 3.1.1 solve problems involving the use and application of the general term formula for arithmetic sequences

3.2 apply the sum formula for arithmetic series, $S_n = \frac{n(a+t_n)}{2}$, $Sn = \frac{n[2a+(n-1)d]}{2}$
- 3.2.1 solve problems involving the use and application of the sum formula for arithmetic series
- 3.2.2 use technology where applicable.

COMMENTS

This concept looks at arithmetic sequences and arithmetic series. There is a wealth of problems and questions to address this concept. Emphasis should be on its application rather than on the use of the formulas for simple calculations.

Sometimes an individual mathematician can be linked directly to a particular topic. This concept is often associated with Carl Friedrich Gauss. A version of the story is related below to provide teachers with an opportunity to discuss the human endeavour in mathematics and to possibly set the stage for some writing by the students about the history of mathematics:

> "When Gauss was about 10-years-old at the first meeting of the arithmetic class, Master Buttner asked the pupils to write down the numbers from 1 to 100 and add them. It was the custom that the pupils lay their slates, with their answers thereon, on the master's desk upon the completion of the problem. Master Buttner had scarcely finished stating the exercise when young Gauss flung his slate on the desk. The other pupils toiled on for the rest of the hour while Carl sat with folded hands under the scornful and sarcastic gaze of the master. At the conclusion of the period, Master Buttner looked over the slates and discovered that Carl alone had the correct answer, and upon inquiry Carl was able to explain how he had arrived at his result. He said, "$100+1=101$, $99+2=101$, $98+3=101$, etc., and so we have as many 'pairs' as there are in 100. Thus the answer is 50×101, or 5050."

(Eves, Howard W., In Mathematical Circles (Quadrants III and IV), Boston: Prindle, Weber & Schmidt, Inc., 1969, p. 112.

In investigating this concept, rather than simply stating a formula and then showing various examples as to how the formula can be used, have the students examine various sequences for their similarities and differences. As well, investigate the terms of a sequence for patterns and if patterns become evident, have the students attempt to formalize the patterns in terms of "n," the number of terms or the position of the particular term in the sequence. Instead of the teacher investigating examples and the students making notes, pose a question and give the students sufficient time to try to solve it themselves. An example may be to do what Master Buttner did and see if you have any "closet Gausses" out there.

PROCESS/PROBLEM-SOLVING CONTEXT

In the solving of application-type word problems, it is important to emphasize to the students that part of "understanding" the problem is to be able to establish the sequences or series to be used.

Problem: Louisa is paid according to the following agreement: she will start at \$5 per hour and at the end of the first week (if she is kept on) she will get a raise of \$1 per hour. Each succeeding week she will get a raise of \$0.25 per hour to a maximum of \$10 per hour. In which week of work will Louisa be earning a rate of pay of \$8.25 per hour?

Solution: The most important thing to emphasize is to set up the sequence. In this case the \$5 is not part of the sequence and must be handled separately. The sequence would be:
6, 6.25, 6.5, 6.75, . . . 8.25.
The arithmetic sequence formula can be used to find "n" and then by adding 1 the students have the answer.

Problem: The formula for the sum of the first "n" terms of an arithmetic series is given by the formula: $S_n = 3n^2 + 2n$. Determine the value of the common difference, d, and the value of the n^{th} term of the original series. Problems such as this send students back to the basics of the meanings of the various formulas. From the given formula, the students can determine that $S_1 = 5$, $S_2 = 16$ and $S_3 = 33$, etc. This leads them to conclude that the first term of the original series is 5, the second term would be $16 - 5 = 11$ and the third term would be $33 - 16 = 17$. Therefore, the common difference is 6 and the value of the n^{th} term is $6n - 1$.

Problem: How many terms of the series $3+7+11\ldots$ are required to yield a sum of 663?

The solution of this problem allows for the development of the formula:

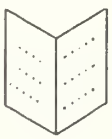
$$S_n = \frac{n[2a+(n-1)d]}{2}$$

which is a combination of the other two formulas of this concept. Most of the text resources discuss its development and use. It not only provides for a solution to this problem but also serves as an alternative method for solving questions where the common difference is known.

TECHNOLOGY INTEGRATION

The use of programmable calculators and computers can be encouraged here as a method of solving problems. The ability of the calculator and computer to "iterate" provides the student with the opportunity to experiment. The emphasis should be on using the technology to do the "leg work" while the student interprets and attempts to apply the answers.

RESOURCE CORRELATION



Holtmath 12:	pp. 421–426
Addison Wesley 12:	pp.
MPP12:	pp. 353–358, pp. 367–371

CONCEPT 4

Students will be expected to demonstrate an understanding that geometric sequences are such that each term is equal to the product of the preceding term and a constant and that a geometric series is the indicated sum of the terms of a geometric sequence.

Students will be expected to:

4.1 apply the general term formula of geometric sequences, $t_n = ar^{n-1}$
- 4.1.1 solve problems involving the use and application of the general term formula for geometric sequences

4.2 apply the sum formula of geometric series, $S_n = \frac{a(r^n - 1)}{r - 1}$
- 4.2.1 solve problems involving the use and application of the sum formula for geometric series
- 4.2.2 use technology where applicable.

COMMENTS

This concept looks at geometric sequences and finite geometric series.

It is important that students be able to distinguish between arithmetic and geometric sequences. Asking students to immediately identify a geometric sequence because of a constant ratio should be encouraged.

Geometric sequences could be introduced using problems involving exponential growth and decay. The teacher could also refer to problems involving the growth of an amount of money invested at compound interest. Students coming from Mathematics 33 will be familiar with this concept. As well, many students have been involved with chain letters of some sort. An interesting problem is to examine the number of letters involved in such a situation. Assuming the chain is not broken, the number of people contacted during the process of the communication forms a geometric sequence.[1]

This section should refer to the exponential and logarithmic functions unit frequently. All the exponential growth and decay problems are essentially geometric sequences or series. Not all growth and decay problems involve doubling or halving. A real effort should be made to have students understand that all problems of this sort involve the following function:

$$A(t) = ig^{t/p}$$

where A is an amount (number, etc.)
t is time
i is the initial measurement
g is the "growth factor"
and p is the growth period for g.

1 See "The Chain Letter: An Example of Exponential Growth," Mathematics Teacher, Volume 80, Number 3, February 1987.

PROCESS/PROBLEM-SOLVING CONTEXT

Although the study of compound interest annuities is not explicitly mentioned, such problems are excellent applications of geometric series. However, care should be exercised as to the difficulty to which students are exposed. The compound period and payment period should be equal. A simple understanding that these topics are examples of geometric series is adequate since in real life these problems are done through the use of tables and computer programs.

Activity 1: Have students calculate the amount of money they would have in their bank account at the end of each year for 25 years if they deposit $100 today and leave it to collect interest at 10% per year compounded annually.

How many years will it take to double the money? Solving exponential equations using logarithms to find the time needed to have a certain investment accumulate to a given amount using the formula $A = P(1+i)^t$ could be reviewed if the unit on logarithms has been done. If students use logarithms, discuss the meaning of a fractional answer and what this means in terms of the stated domain.

Activity 2: Have students calculate the amount that will accumulate in a savings account if $100 is invested yearly for a certain period at 10%/a compounded annually. Compare the amount with Activity 1. Draw graphs of the amounts of each investment over time. How much of the money in the account is interest?

Activity 3: The job of washing windows is offered at your school. There are 35 windows to be washed. There were two methods of payment offered:

a. $8 per window

b. 1 cent for the first window, 2 cents for the second window, 4 cents for the third window, 8 cents for the fourth window, 16 cents for the fifth window, and so on. Which method of payment would you prefer? Most students will immediately choose plan a. Have students prepare a table similar to the one below.

Window	Price/b. Window	Total b.	Total a.
1	0.01	0.01	$8
2	0.02	0.03	$16
3	0.04	0.07	$24
4	0.08	0.15	$32
5	0.16	0.31	$40
•			
•			
•			

The article "Geometric Growth and the Hand-Held Calculator," Mathematics Teacher, Volume 73, Number 8, November 1980, presents this problem and discusses further activities for introducing the student to geometric series and their sums.

TECHNOLOGY INTEGRATION

Spreadsheets can be used to see how your money builds in the activities above.

The use of calculators is recommended, particularly for the activities above, because of their iterative ability in such calculations.

See Holtmath 12, p. 454 for a computer application involving the sum of a geometric series.

EXTENSION

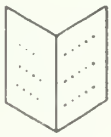
This concept could be extended to infinite geometric sequences and series and to an extended study of compound interest and annuities. These topics can be found in Holtmath 12, pp. 434–451, in MPP12, pp. 377–396 and in AW30, pp. 361–364.

Two articles in Mathematics Teacher on geometric series may be of interest.

1. "Visualizing the Geometric Series," Volume 82, Number 2, February 1989.

2. "A Geometric View of the Geometric Series," Volume 77, Number 6, September 1985.

RESOURCE CORRELATION



HM12:	pp. 427–431
MPP12:	pp. 359–363
	pp. 372–376
AW30:	pp. 339–344
	pp. 356–359

MATHEMATICS 33

PROGRAM EMPHASIS

Required Content

The major part of the content of each high school mathematics course consists of topics required of all students who take the course. The **required content** comprises 80% of the course and contains the concepts, skills and attitudes that all students are expected to acquire. As well, the required portion of all courses includes specific expectations in regard to problem solving and the use of technology.

Elective Material

Each course will include a compulsory component comprising 20% of the course, made up of **elective material** that will be consistent with the content and expectations of their required component. The purpose of the elective material is to provide for enrichment, remediation, or innovative experimental presentations or activities. It is not intended to provide acceleration or advanced placement. However, horizontal enrichment and extension is appropriate and students should have access to elective material that serves their individual needs and interests.

Suggested Program Emphasis (in %) for Required Portion of Mathematics 33

The following are suggestions only to guide teachers in planning. Individual students may require more or less effort to successfully complete a particular unit of study.

Unit	Suggested Program Emphasis (in %)
Radicals and Exponents	5
Annuities, Mortgages and Loans	5
Statistics	15
Trigonometry	14
Polynomials and Rational Expressions	8
Functions and Relations	16
Quadratic Functions and Relations	17
Elective	20

ATTITUDES

Students will be expected to demonstrate an attitude associated with mathematical literacy. In particular, *students will be expected to*:

- be confident in their mathematical knowledge and in their ability to acquire new knowledge;
- demonstrate persistence, resolve, flexibility and ingenuity in finding the solution to problems;
- develop intellectual curiosity and openness to new ideas, insights and change in the pursuit of mathematical knowledge;
- exhibit an attitude of curiosity and spontaneity and appreciate creativity and innovation in representing situations mathematically;
- be critical and constructive in approaching new ideas and new processes;
- be aware of the importance of communication skills in mathematics;
- appreciate the usefulness of computational competence, mathematical processes and problem-solving skills that are used in the decision-making and modelling processes in our society;
- appreciate the contributions of mathematics to our culture and civilization.

PROBLEM SOLVING

Students will be expected to use a variety of procedures to help them understand mathematical problems. In particular, they will be expected to:

- read the problem thoroughly
- identify and clarify key components
- restate the problem using familiar terms
- evaluate the given information as to whether it is insufficient or extraneous
- interpret pictures, charts and graphs
- determine any hidden assumptions
- ask relevant questions
- identify given, needed and wanted information
- diagram or model the problem situation
- use suitable notation
- determine valid inferences.

Students will be expected to develop a variety of strategies for use in the solution of mathematical problems. In particular, they will be expected to:

- conduct an investigation
- use estimation and approximation
- develop equations or use formulas
- use flow charts
- make lists and charts
- look for patterns
- work backward
- break the problem into smaller parts
- look for a simpler or related problem
- make diagrams or models
- use manipulatives
- choose and sequence a series of mathematical operations

- sketch the graph of a problem situation
- establish procedures to gather and organize data.

Students will be expected to develop a variety of skills that can be used to carry out the plan for the solution of a problem. In particular, they will be expected to:

- apply selected strategies.
- present ideas clearly
- document the solution process
- use appropriate group behaviours
- use calculators and computers
- evaluate problem-solving strategies for effectiveness
- search for additional information
- ask questions
- be open to inspirations, intuitions and "bright ideas."

Students will be expected to employ a variety of skills to help them look back over the solution of a problem. In particular, they will be expected to:

- determine the reasonableness of an answer
- explain the solution in oral or written form
- consider the possibility of additional solutions
- search for other strategies and processes of solution
- create and solve similar problems
- note the characteristics that will be identifiable in similar problems
- make a generalization.

RADICALS AND EXPONENTS
SUGGESTED PROGRAM EMPHASIS – 5 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that radical expressions and equations can be simplified by performing basic operations on numbers and expressions written in radical form.	Students will be expected to change the form of second and third order radicals from mixed to entire and vice versa.	
	Students will be expected to perform the operations of addition, subtraction, multiplication and division involving second order radicals.	Students will be expected to rationalize radicals with monomial denominators involving one second order expression.
	Students will be expected to solve second order radical equations.	Students will be expected to solve problems involving radical equations.

CONCEPT 1

Students will be expected to demonstrate an understanding that radical expressions and equations can be simplified by performing basic operations on numbers and expressions written in radical form.

Students will be expected to:

1.1 change the form of second and third order radicals from mixed to entire and vice versa

1.2 perform the operations of addition, subtraction, multiplication and division involving second order radicals

1.2.1 rationalize radicals with monomial denominators involving one second order expression

1.3 solve second order radical equations

1.3.1 solve problems involving radical equations.

COMMENTS

It may be useful to have a quick review of the Mathematics 23 exponent work before the introduction of radicals. During the review of exponent work it is a good time to show the use of "fractional" calculators to assist in the solution of integral based fractional exponents (ex. $16^{\frac{1}{4}}$). Many of these fractional calculators are capable of transforming improper fractions to mixed fractions and to decimal forms. This idea may be useful in the introduction of entire radicals to mixed radicals. Performing the operations in this radical unit might be linked with operations of other number systems. Linking the operations of radicals with previous operations may help students retain these "new" properties.

Note: Use of one radical within the equation is recommended. If more than one radical is used the resulting quadratic equation should be solved by the factoring method.

It may be of some use to show the students how parallel the operations in previous units are with operations in this unit.

Similarities between radicals and other units.

	Entire to mixed radical	Improper fractions to mixed fractions power and rational laws of exponents
1.1	$\sqrt{125a^5}$	$\frac{128a^5}{24a^3}$
	$\sqrt{25}\times\sqrt{5}\times\sqrt{a^4}\times\sqrt{a}$	$\frac{8\times16\times(a^3)\times(a^2)}{8\times3\times(a^3)}$
	add/subtract radicals	polynomial "like" terms

1.2	$5\sqrt{3a}-3\sqrt{2b}+7\sqrt{3a}-4\sqrt{2b}$	$5x^2+7x-4x^2+12x$
	multiplication and division	similar polynomial rules
1.2	$(5\sqrt{2a}+3\sqrt{b})(3\sqrt{5a}-2\sqrt{3b})$	F.O.I.L.
		$(3x+2y)(5x-6y)$
		eliminating common factors
1.2	$\frac{18\sqrt{14a^6}}{6\sqrt{7a}}$	$\frac{24a^4b^5}{16a^3b^2}$
	solving radical equations	solving simple quadratic equations
1.3	$\sqrt{5a+1}-16=0$	$8x^2-10=22$

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1.3: For what value x does $\sqrt{2x}=x\sqrt{2}$?

Solution: Square both sides $2x=2x^2$

subtract 2x $0=2x^2-2x$

factor 2x $0=2x(x-1)$

solve for x $x=0, x=1$

verify solutions:

$x=0\sqrt{2(0)} \mid 0(\sqrt{2})$; $x=1\ \sqrt{2(1)} \mid 1(\sqrt{2})$

$0 \mid 0$; $\sqrt{2} \mid \sqrt{2}$

$\{0\}$; $\{1\}$

solutions $\{0,1\}$

Problem 1.3: Solve for x (verify answers) $3+\sqrt{3x+1}=x$

$\sqrt{3x+1}=x-3$

$(\sqrt{3x+1})^2=(x-3)^2$

$3x+1=x^2-6x+9$

$0=x^2-9x+8$

$0=(x-1)(x-8)$

$x=1, x=8$

with verifying: $x=1$ is extraneous; solution $\{8\}$.

Problem 1.3.1: Herons' formula for the area for any triangle involved using the semi-perimeter "s" and the length of the three sides a, b, c: $A=\sqrt{s(s-a)(s-b)(s-c)}$. With a triangle of sides 14 cm, 16 cm and 24 cm, determine the area in radical form.

Solution: $s=\frac{14+16+24}{2}$ $\quad s=27, a=14, b=16, c=24$

$A=\sqrt{s(s-a)(s-b)(s-c)}$

$A=\sqrt{27(27-14)(27-16)(27-24)}$

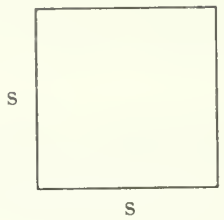
$A=\sqrt{11583}$

$A=\sqrt{81\times143}$

$A=9\sqrt{143}$

Problem 1.3.1: For a square the area is found by: $A = s^2$. Using Pythagorean Theorem a diagonal $d = ?$

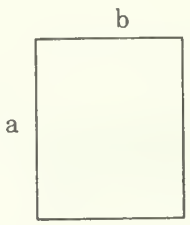
A square with an area of 64 sq. units would have sides and diagonals of what dimensions?



Solution: $(s = 8, d = 8\sqrt{2})$

Problem 1.3.1: For a rectangle the area is found by: $A = a \times b$. Using Pythagorean Theorem a diagonal $d = ?$

If the are a is 30 sq. units and one side is $3\sqrt{2}$, determine the lengths of the missing side and the diagonal.



Solution: $(5\sqrt{2}, d = 10)$

The following problems offer a third root and a combination of radical terms:

Problem 1.3: Extension solve for x (verify answers)
$\sqrt[3]{7x-6} = 4$

Solution: $\{10\}$

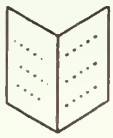
Problem 1.3: Extension solve for x (verify answers)
$\sqrt{2x+5} = 2\sqrt{2x+1}$

$$
\begin{aligned}
(\sqrt{2x+5})^2 &= (2\sqrt{2x}+1)^2 \\
2x+5 &= 8x+4\sqrt{2x}+1 \\
(-6x+4)^2 &= (4\sqrt{2x})^2 \\
36x^2-48x+16 &= 32x \\
36x^2-80x+16 &= 0 \qquad \text{Verifying } x=2 \text{ is extraneous} \\
9x^2-20x+4 &= 0 \\
(9x-2)(x-2) &= 0 \qquad \text{solution } \{\tfrac{2}{9}\ \} \\
x = \tfrac{2}{9},\ x &= 2
\end{aligned}
$$

TECHNOLOGY INTEGRATION

Use of fractional calculators, graphic calculators and computer spreadsheets are becoming incorporated into many areas of mathematics courses. This unit could use all three technological tools and would enhance the variety of learning styles and techniques in the classroom.

RESOURCE CORRELATION MMW4: pp. 5–6, pp. 18–33; 36–38; 92–96



ANNUITIES, MORTGAGES AND LOANS
SUGGESTED PROGRAM EMPHASIS – %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that an annuity is a sequence of payments paid in regular intervals.	Students will be expected to calculate the amount and present value of an annuity using tables or calculators.	Students will be expected to solve simple problems involving the amount and present value of an annuity. Students will be expected to use a calculator or computer spreadsheet to help calculate the amount and present value of an annuity.
Students will be expected to demonstrate an understanding that the repayment schedule on loans and mortgages is determined by the amount borrowed, the interest rate, the number of payments and the total time over which the repayment occurs.	Students will be expected to differentiate between a mortgage and other types of loans. Students will be expected to calculate mortgage and loan payments using tables.	Students will be expected to solve problems involving the calculation of payments on loans and mortgages. Students will be expected to use a calculator or computer to help in the solution of problems involving loans and mortgages.

CONCEPT 1

Students will be expected to demonstrate an understanding that an annuity is a sequence of payments paid in regular intervals.

Students will be expected to:

1.1 calculate the amount and present value of an annuity using tables or calculators
- 1.1.1 solve simple problems involving the amount and present value of an annuity
- 1.1.2 use a calculator or computer spreadsheet to help calculate the amount and present value of an annuity.

COMMENTS

Teachers must note carefully the way in which the concept is worded. Unlike Mathematics 30, the emphasis in Mathematics 33 is on the use of tables and not on the development of formulas. The focus of this treatment, then, should be on discussing the concept of an annuity versus that of a lump sum ($A = P(1+i)^n$, which was studied in Mathematics 23) and the difference between amounts and present values. In the Mathematics 13–23–33 sequence, the emphasis is often on the practical usefulness of the concepts with some of the theoretical underpinnings emphasized less. Hence, the emphasis here is more on the use of tables and the understanding of the problem than in the substitution into a formula.

Very few topics have the capacity to attract attention like "money." Teachers can capitalize on this potential to discuss several related concepts such as the use of tables, problem-solving strategies and estimation as well as the role of technology. There is an opportunity to generate realistic scenarios that could directly affect students (especially those students who have part-time jobs and are saving for specific purposes) and use these ideas to guide the investigations.

The work on compound interest from Mathematics 23 is a good place to start. This review will provide an initial understanding of the origin of the numbers in the tables.

Students have difficulty distinguishing between "amount" and "present value" situations. Teachers are well advised to address this problem. One way to view "amount" questions is as "saving" situations (i.e., deposits are made to accumulate to a total at the end of a specified period) and "present value" questions are "spending" situations (i.e., withdrawals are made over a specified period from a total available at the start of the period).

PROCESS/PROBLEM-SOLVING CONTEXT

To provide some interest, you might consider starting the unit with a problem that is sufficiently complicated that the students cannot immediately see the solution, and that may have some relevance to them. For example, consider the following:

Problem: How much money would you like to have available to you to supplement your pension plan in your retirement? Once you have arrived at an annual amount, how long do you think that you would want this money available during your retirement? How much would you have to save each year that you work to be able to fund your retirement plan?

Solution: Assume an annual retirement amount of $30 000, a retirement period of 20a, an annual interest rate of 10% and a working period of 30a. By working backward, the present value of the retirement annuity is approximately $255 406. The annual amount required to generate this total over 30a is approximately $1552.68.

Another way to start this unit is to have the students do some financial planning that may include annuities, mortgages, loans, etc. This could be formulated as a project (group or individual) that might grow into class presentations. There are many exciting things to consider. Studying the unit will provide the knowledge framework to assist the students in their work.

To investigate the use of tables, consider the following problems:

Problem 1.1.1: Thomas plans to save $1000 in each of the next 25 years and to invest it at 10%/a compounded annually. How much will he have at the end of that time?

Solution: The first thing that students should note is that in most resources there will be two tables that refer to "amount" and two tables that refer to "present value." Students should be directed to use the tables that also have "annuity" in the title. In this case, they should use the "Amount of an Annuity" table as Thomas is saving money to accumulate a total at the end of the 25 years. The number they will need for the solution will be the intersection of the 10% column (across the top) and the row marked "25" (down the side). This number is 98.347 06. The final solution is 1000 (98.347 06) = $98 347.06.

Problem 1.1.1: If Thomas planned to deposit his money in two half-yearly (semi-annually) deposits of $500 each for the 25 years, what would be the total if the interest was 10%/a compounded semi-annually?

Solution: Students should now be reminded of the adjustments they studied in Mathematics 23 when the interest term is other than annual. The interest rate (10%/a) should be halved and the number of payments should be doubled. Therefore, the students need to find the intersection of the column headed 5% and the row marked 50. This number is 209.348 000, which is multiplied by the value of each deposit, $500, to yield a result of $104 674.00.

Problem 1.1.1: Suppose Thomas' great uncle Dudley had set up a trust account that would pay Thomas $1500 a month for four years. What amount would Uncle Dudley require to generate this annuity if no further contributions were to be made assuming interest at 12%/a compounded monthly?

Solution: Since this is a spending scenario, students should be directed to the "Present Value of an Annuity" table. Once again an adjustment must be made with the interest rate and the period to coincide with the compounding period. The number that the students will require is the intersection of the "1%" column and the "48" row, which is 37.973 96. The final answer is 1500(37.973 96), which is $56 960.94. A significant point for students to note is that Thomas will receive a total of $72 000 over the four years.

A potential question for an exam or quiz is to ask why there is a difference between the amount that Thomas will collect and that Uncle Dudley will put into the fund. Also ask that they substantiate their answer with some calculations.

Problem 1.1.1: Suong would like to accumulate a total of $10 000 four years by making equal annual deposits. If the rate of interest is 9%/a compounded semi-annually, how much should she deposit each time?

Solution: This question allows the students the opportunity to explore some other possibilities using a similar analysis. Since this is a "saving" scenario, the "Amount of an Annuity" table is used with a column heading of "4.5%" and the row "8." The intersection is 9.380 00. However, instead of multiplying, students should realize that division should be used as this time the total is known and it's the size of the deposit that is to be determined. The answer is $1066.10.

Students should be encouraged to develop, discuss and try many of their own situations to practise this concept. The development of student-generated questions along with their solutions could also become part of an evaluation scheme.

The text resources offer ample opportunities for students to learn the basics of this material. However, teachers should explore some situations that may require the students to expand their thinking. This should be done keeping in mind the capabilities of the students and the philosophy of this sequence. Consider this next problem as an example.

Problem 1.1.1: Jacob would like to do some financial planning for an extended family trip. He would like to know how much he should invest annually to generate an income of $2000 per month for two years and he is prepared to work (and save) for 10 years before the trip. He assumes that interest should be 12%/a compounding annually during the working (accumulating) years and compounding monthly during the trip years.

Solution: Questions like this should give the students an opportunity to use their problem-solving skills. It combines both a present value of an annuity and an amount of an annuity as well as requiring both multiplication and division calculations. One solution is to first find the present value of the 24 $2000 payments, which is $42 486.78. This becomes the desired total for the accumulation phase. Using this total, then, the amount of each annual deposit is $2421.09. Even though questions like this require that students be conversant with all of the material in this concept, it is a complicated question and should be balanced with a number of straightforward ones. You should try to achieve a balance between routine and challenge for those who can benefit from it.

Retirement scenarios (like the one posed above) are very interesting to investigate as students are often amazed at the effect of compound interest over 30, 40 or even 50 years. The problem is that many resources do not have tables that will cover such a wide time span. However, it might be an interesting project for students to generate extended tables using spreadsheets.

TECHNOLOGY INTEGRATION

If the focus of this treatment is on tables, the use of calculators will be somewhat restricted to calculations. However, programmable calculators could be used to some extent to accept values obtained from a table and calculate the projected totals. Such rapid calculations then allow the students to hypothesize about the effect of varying such factors as compounding period and interest rates. An interesting project may be to visit a local financial institution, survey the savings options available and then determine possible outcomes for various savings scenarios. Some students may have access to special financial calculations and software for computers that could serve as a rich resource for further investigation in the classroom (a high school "show and tell").

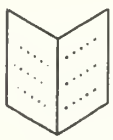
Investigations such as those mentioned above could profit from the use of a spreadsheet. Of course, spreadsheets require the use of formulas, but the formulas required can be presented as formulas for this particular application without the need for development. This is another way for students to generate a significant number of data from which they can draw some conclusions. Students should also be encouraged to talk to other resource people about the use of spreadsheets and about other aspects of this material. Such potential resource people include the business education department in their own school as well as other private and public institutions in the community.

EXTENSION

Some questions, such as the number of payments/deposits, may require the use of specialized formulas like those from geometric sequences and series. These may be legitimate extensions. However, caution should be used when such formulas are used out of context. This should be viewed solely as an extension after the facility with tables has been well established. Students should not be expected to memorize any formulas used for this purpose.

With a background in compound interest established in this unit, students might be intrigued with the discussion on the "Rule of 78," presented in Mathematics Teacher, September 1988, pp. 450–453.

RESOURCE CORRELATION MM4: pp. 289–302 (pilot edition)



CONCEPT 2

Students will be expected to demonstrate an understanding that the repayment schedule on loans and mortgages is determined by the amount borrowed, the interest rate, the number of payments and the total time over which the repayment occurs.

Students will be expected to:

2.1 differentiate between a mortgage and other types of loans
2.2 calculate mortgage and loan payments using tables
  2.2.1 solve problems involving the calculation of payments on loans and mortgages
  2.2.2 use a calculator or computer to help in the solution of problems involving loans and mortgages.

COMMENTS

As was stated in the previous concept, few ideas have greater capacity to generate interest in students than "money," although students may be more interested in the "bank loan" section than in the one on "mortgages." This concept also provides an opportunity for some consumer education. Agencies like Alberta Consumer and Corporate Affairs are a rich resource to supplement the text material and may also provide material as a focus for student research and projects.

The differentiation between mortgages and bank loans will be a more qualitative discussion than some of the other concepts. Some of these differences include facts such as the cost involved in setting up a mortgage versus a bank loan. This is necessitated by the fact that a mortgage is registered with the land titles office and as such requires the services of a lawyer. As well, mortgages are generally used for larger amounts of money and amortized over longer periods.

It is important to remember that the focus is on the use of tables, not formulas. In addition, questions should be kept relatively basic as this will be a first look at this material for most students. The emphasis should be on good understanding of the basics rather than marginal understanding of more complicated material.

PROCESS/PROBLEM-SOLVING CONTEXT

Once again, students themselves can be a rich resource for possible examples and problems. A good place to start would be a small bank loan to buy some article of interest like a used car or a motorcycle. The texts have ample questions to give students the opportunity to become proficient at the calculations. The real power of the unit is the way in which it lends itself to some of the extras that mathematics students don't often get a chance to do.

Activity 2.1: Arrange for a field trip to a local financial institution and have an officer explain the difference between a mortgage and a loan for personal articles like a car or a vacation.

Activity 2.2: Students could do a research project compiling some data about the advantages and disadvantages of buying a home with a large down payment or a small one, looking into the advantages of paying down the principal with some extra cash or comparing the options of buying a home versus renting and investing. This could be a great opportunity for a spreadsheet.

There are many options available to students and teachers as to how this concept can be studied and evaluated. Among those could be student projects such as the ones mentioned in Concept 1 in which students look into developing a financial plan that might encompass some long- and short-term planning. There are probably many resource people among their parents and/or other members of the community who are employed as controllers of companies, accountants, bankers and other members of the financial community who would more than likely be pleased to assist in exploring this unit.

There may also be some chances for some cross-discipline studies. For example, some enterprising young programmers in Computer Processing courses could be writing programs for amortization tables or a ledger showing the schedule of payments to retire a personal loan over a given period. There might also be some opportunity to look at some of the options available in the Accounting courses.

TECHNOLOGY INTEGRATION

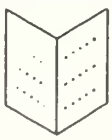
Since much of the basic information comes from tables, the calculator becomes just a tool for calculation. However, a programmable calculator or a spreadsheet application for a computer could provide students with the capability to investigate many different possible options to a problem. Again, it is important to keep in mind that formulas should be used only as an extension and not as core material.

A spreadsheet offers the opportunity for many different comparisons. For example, it can provide information for a comparison between rates and payment options. Students could see, investigate and comment on perceived similarities and differences. Such analysis could lead to some interesting projects.

EXTENSION

The basic text resources offer the potential for enhancement of some of this material. Such enhancement may involve the formula for geometric series (especially in some spreadsheet applications). Once again, teachers are cautioned that such extension should be offered once the student has demonstrated competence with the basic material.

RESOURCE CORRELATION MM4: pp. 303–305 (Alberta pilot edition)



STATISTICS
SUGGESTED PROGRAM EMPHASIS – 8 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that a bivariate distribution involves two variables that may have some relationship to each other.	Students will be expected to plot sets of bivariate data on a scatter plot. Students will be expected to plot a line of best fit on a scatter plot using the median fit method. Students will be expected to develop and use prediction equations of the line of best fit to make inferences for populations. Students will be expected to determine the apparent correlation between the variables of a bivariate distribution from a scatter plot, including the strength and direction of the correlation. Students will be expected to collect, organize and analyze sets of bivariate data.	Students will be expected to apply statistical processes and statistical reasoning in investigations involving bivariate data.
Students will be expected to demonstrate an understanding that the results of a survey can be interpreted with measurable degrees of confidence.	Students will be expected to distinguish between a population and a sample and assess the strengths, weaknesses and biases of given samples. Students will be expected to collect and organize the results of yes/no surveys taken from defined samples. Students will be expected to draw box plots of the results of multiple samples. Students will be expected to use charts of 90% box plots to find the confidence interval within which a survey result can be interpreted. Students will be expected to draw statistical conclusions, to make inferences to populations, and to explain the confidence with which such conclusions and inferences are made based on the results of yes/no surveys.	Students will be expected to design and administer a simple survey. Students will be expected to collect and organize the results of a simple survey. Students will be expected to carry out investigations involving multiple samples taken from populations with known and unknown proportions of yeses. Students will be expected to use statistical inferences to solve problems based on 90% confidence intervals. Students will be expected to design and administer a survey to a random sample of a population, collect and organize the responses, and analyze the results, including making inferences to the population and evaluating the results for the confidence with which they may be held.

CONCEPT 1

Students will be expected to demonstrate an understanding that a bivariate distribution involves two variables that may have some relationship to each other.

Students will be expected to:

1.1 plot sets of bivariate data on a scatter plot
1.2 plot a line of best fit on a scatter plot using the median fit method
1.3 develop and use prediction equations of the line of best fit to make inferences for populations
1.4 determine the apparent correlation between the variables of a bivariate distribution from a scatter plot, including the strength and direction of the correlation
1.5 collect, organize and analyze sets of bivariate data
    1.5.1 apply statistical processes and statistical reasoning in investigations involving bivariate data.

COMMENTS

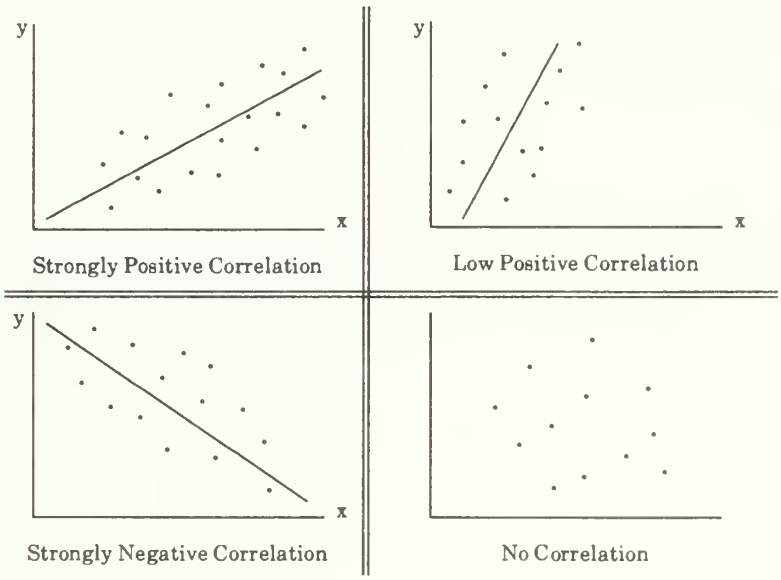
Students have worked with univariate data in Mathematics 13. These were situations where one variable is measured for a given group; for example, the heights of students in a class. The techniques and skills used include the representation of data by line plots, stem and leaf plots, box plots and calculating various averages. This concept involves situations where two variables are measured for each individual in a group; for example, the height and circumference of the neck are measured for each class member. The data collected on each student can be considered a point on a Cartesian plane. The resulting graph is called a scatter plot. The scatter plot is a useful technique for investigating the relationship between two variables. Some relationships may be approximated with a "best-fit" line, in which case the equation of the line should be found. The line and equation can be used to make predictions. The limitations of such a procedure should be mentioned, especially when the data are extrapolated.

There are different ways to plot a line of best fit. The common method described in texts is the median method. Many calculators will calculate the slope and y-intercept of the best line using the "least squares" method. Some spreadsheets will do this type of calculation as well. It is important to remember that calculating numbers is not the focus of this unit. Students should be able to determine the equation of this line once they know the slope and y-intercept. Also important to note is that a straight line may not be appropriate for all types of data. The relationship may be curvilinear. The data could be clustered into two groups where it may be appropriate to draw a different straight line for each cluster.[1]

1. Landwehr, James. Exploring Data: Quantitative Literacy Series. p. 133.

The strength and direction of the relationship between the variables can be summarized with a number called the correlation coefficient. If all the points fall perfectly along a line with positive slope then the correlation coefficient is +1. If the points exactly form a line of negative slope then the correlation coefficient is –1. A value of zero for the correlation coefficient means that no relationship exists between the variables. In most cases the correlation will be a value between –1 and +1. A positive coefficient indicates a positive relationship, meaning that as one variable increases, so does the other. A negative correlation coefficient indicates a negative relationship where one variable increases as the other decreases. Students should not be required to calculate the coefficient unless software exists that does this easily. A qualitative understanding of this value is much more significant.

Correlation in no way implies causality. There are many reasons why two variables may correlate highly. For example you may find a positive correlation between the amount of math homework a student does and their positive feelings toward mathematics. However, it may not make sense to assume that more homework will give a student more positive feelings toward the subject. For a more extensive discussion of this topic see Campbell's book Flaws and Fallacies.[2]



2. Campbell, Stephen. Flaws and Fallacies in Statistical Thinking. Prentice Hall (1974), p. 152.

The NCTM Standards document[3] suggests a number of things that are applicable to this unit. Student activities should focus on asking questions about their environment. Examples given should involve real, not contrived, data. Projects must be an integral part of the learning and evaluation process. The focus must **not** be on the ability to calculate a particular value or values, but rather on analyzing and communicating numbers and ideas.

An excellent way to begin study of this concept is to ask students to suggest variables that might be related in positive and negative ways. A positive relationship might exist between hours of study and marks on tests or the height of a father and his son. A negative relationship might exist between the number of hours of "after-school" employment and marks in school. Students should be given an opportunity to investigate these relationships by collecting data and making scatter plots.

PROCESS/PROBLEM SOLVING CONTEXT

Activity 1: Have students look in almanacs or statistical yearbooks and collect information about various countries. From the abundance of data they may have a theory about two variables that are related (positively or negatively).

e.g., rainfall and GNP
population and pollution
area and population

Investigate this relationship and prepare a presentation. This would also be an excellent cooperative project.

Activity 2: Have students ask the principal for attendance figures on various days. Phone the weather office and ask for the average temperature on those days. Do they correlate? Can you predict the attendance if you know the temperature?

Activity 3: Sports fans have access to an abundance of statistics. Every day the sports section of the newspaper is filled with opportunities for investigations of relationships between variables. Statistical yearbooks that summarize last year's performance are available on newsstands.

e.g., hockey – shorthanded goals scored: for and against
goals scored: for and against
football – yards of offence and points scored
baseball – runs scored and number of hits

Activity 4: From the beginning of the year ask students to keep an accurate log of the amount of work (in hours) they do in each unit. Attempt to correlate with their marks on these units.

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3. National Council of Teachers of Mathematics. Curriculum and Evaluation Standards for School Mathematics, Preston, VA, p. 105.

Activity 5: Present the students with a statistical argument that will predict their final mark in the course. For example, by keeping records of students' performance in Mathematics 23 and Mathematics 33 from previous years, the current students could predict their final mark in the course. Have the students critique this argument.

Activity 6: Have students collect univariate data on each member of the class. It is important to keep track of the student's name with each value collected. A wide variety of data is best.
Examples:
- the mark on a spelling test designed by the experimenter
- the number of free throws out of 10 made on a nerf basketball hoop
- how many of the Seven Dwarfs' names can be recalled
- the accuracy of a guess of the number of jelly beans in a jar
- the number of albums, C.D.s or cassettes owned.

Each student should summarize their findings using the techniques for univariate date. The data and summaries could be posted around the room. Students are then asked to investigate two possible relationships using this data.

1. A relationship that they believe has a highly positive or highly negative correlation coefficient (i.e. close to either +1 or –1). For example the mark on the spelling test and the number of "Seven Dwarfs" recalled may correlate highly.

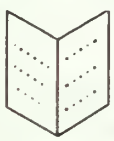
2. A relationship that they believe has a very small correlation coefficient (i.e. close to zero).

TECHNOLOGY INTEGRATION

Scientific calculators and spreadsheets are designed to specifically handle the calculation of correlation coefficients and equations of "best-fit" lines. Graphing calculators and some spreadsheets will easily produce a scatter plot of the data.

RESOURCE CORRELATION

MMW: pp. 296–302
MM4: pp. 251–258



CONCEPT 2

Students will be expected to demonstrate an understanding that the results of a survey can be interpreted with measurable degrees of confidence.

Students will be expected to:

2.1 distinguish between a population and a sample and assess the strengths, weaknesses and biases of given samples

2.2 collect and organize the results of yes/no surveys taken from defined samples
- 2.2.1 design and administer a simple survey
- 2.2.2 collect and organize the results of a simple survey

2.3 draw box plots of the results of multiple samples
- 2.3.1 carry out investigations involving multiple samples taken from populations with known and unknown proportions of yeses

2.4 use charts of 90% box plots to find the confidence interval within which a survey result can be interpreted
- 2.4.1 use statistical inferences to solve problems based on 90% confidence intervals

2.5 draw statistical conclusions, make inferences to populations and explain the confidence with which such conclusions and inferences are made based on the results of yes/no surveys
- 2.5.1 design and administer a survey to a random sample of a population, collect and organize the responses, and analyze the results, including making inferences to the population and evaluating the results for the confidence with which they may be held.

COMMENTS

This concept involves making and interpreting surveys. An appropriate introduction would be to show students examples of surveys from the media. There are various types of surveys but this concept deals only with "Yes-No" surveys.

In this type each person asked must respond with "Yes" or "No." Examples of acceptable "Yes-No" survey questions would be:

a. Did you have an average over 80% last year?
b. Do you like rap music?

Examples of questions that would be unacceptable for "Yes-No" surveys would be:

a. What was your average last year?
b. React to the statement below by circling a response.
"I love rap music."

strongly agree	agree	neutral	disagree	strongly agree

Surveys are an integral part of inferential statistics. The goal is to infer something, usually the mean, of a large group, called the population, by collecting information from a subset of this group, is called a sample. One of the problems is that a random sample does not always accurately reflect the population (see Activity 1).

A survey with a sample size of 20 could select eight smokers at random. How accurate is the conclusion of this survey that 40% of the population are smokers? (i.e. 8 out of 20 = 40%) If the surveyor were to say that the mean is 40% ± 10% in nine out of 10 cases, the reader of the survey would have a better indication of the limitations of the information. This "additional" information is called the 90% confidence interval of the mean. It reveals that if the survey were conducted 10 times, nine out of 10 times the mean number of smokers in the survey would be between 6 and 10 (i.e. between 30% and 50% if the sample size is 20 people). One of the major problems to be tackled in this concept is the establishment of confidence intervals.

The establishment of confidence intervals requires some backtracking. The focus will be on a population that has a known number of "yeses." In a room filled with eight smokers and 12 nonsmokers, a random selection of 10 names from a hat will not always give the same proportion of smokers and nonsmokers. However, some sample proportions are unlikely to occur.

A simulation is an effective method to begin discussion of confidence intervals. A random number table can be used to simulate taking samples of size 10 from a known population of 40% smokers. If the digits 0, 1, 2 and 3 represent "yeses" and all others represent nonsmokers, many samples can be taken in a short time while using the table. Several textbooks also list computer programs that will do thousands of samples in minutes.

Here is an example of a computer simulation of a population of 40% smokers where the sample size is 10. The results of this simulation will vary in detail each time it is done but the major features are constant if enough trials are completed. The column labelled "likely" or "unlikely" is discussed below.

Population percentage of yeses = 40%
Sample Size = 10
Number of samples analyzed = 1000

number of yeses	frequency	sample proportion	likely or unlikely
0	3	0.00	unlikely
1	45	.10	unlikely
2	115	.20	likely
3	224	.30	likely
4	240	.40	likely
5	195	.50	likely
6	129	.60	likely
7	35	.70	unlikely
8	13	.80	unlikely
9	1	.90	unlikely
10	0	1.00	unlikely

Notes:

- Sample proportion $= \frac{\text{no. of yeses}}{\text{sample size}}$

- The frequency of each class can be predicted using the Binomial Distribution formula because the "Yes-No" Survey is a binomial experiment. Students need not ever use this formula but it may be useful for the instructor to check the simulations.

  $$\text{frequency} = N \times \frac{n!(p^y) \times (1-p)^{n-y}}{y!(n-y)!}$$

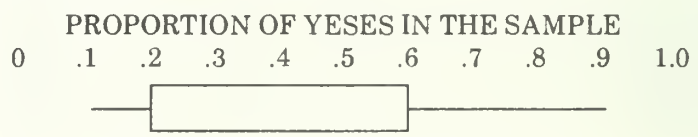
  where N = number of samples taken, n is the size of each sample, y is the number of yeses in the sample and p is the population percentage of yeses.
  e.g.,
  For our simulation N = 1000, n = 10, p = .40 and an arbitrary choice of y is 7. Therefore:

  $$\text{freq.} = \frac{1000 \times 10!(.40^7)(.60)^3}{7!3!} = 42.26$$

- Tables are readily available for the binomial distribution. Most will list the probability of a certain number of yeses within each sample size.

The curriculum focuses on 90% confidence intervals. To do this, 90% of the values (900 out of the 1000 in this simulation) are labelled as likely to occur. As a result, 5% of the values at either end are labelled as unlikely. For the simulation above 0, 1, 7, 8, 9 and 10 are all labelled as unlikely numbers of yeses. The total frequency of these values is '3+45+35+13+1+0' or 97. This represents about 10% of the simulations that were done. The total frequency of the other values is 902, which is about 90% of the simulations that are done. Therefore, for a population of 40% yeses and a sample size of 10, you would expect to find the number of yeses in a sample to be between 2 and 6 about 90% of the time.

A box plot can be used to summarize this table. The box would contain 90% of the values and the whiskers 5% each. These are called 90% box plots. The whiskers should never contain more than 5% of the values. The sample proportions that are likely are therefore inside or on the borders of the box. For the simulation done the likely proportions are between 0.2 and 0.6 using 90% confidence intervals.



The textbooks all contain tables of 90% box plots for various sample sizes. These are box plots, like the one above, for populations that vary from 0% to 100% yeses. A class project could be to "build" one of these tables.

Students' previous experience with box plots will be with ones that use the median and are therefore 50% box plots. This means that the box contains 50% of the values while each whisker has 25% of the values.

A chart of 90% box plots is used to establish 90% confidence intervals for surveys. If for example you conduct a survey with a sample size of 20, you would refer to the chart with that size. If you recorded eight "yeses" in your sample of 20, your sample proportion is 40%. However you want to know the population percentage of "yeses". A sample proportion of 40% yeses is likely to occur in a wide variety of populations. By finding .4 on the top row of the chart and drawing a vertical line it can be seen that .4 is a likely sample proportion in populations that have from 25% to 60% "yeses." Therefore, the results of the survey should be reported as 40% $\pm$15% in nine out of 10 cases.

It is important to note the effect of increasing the sample size from, for example, 20 to 40. If 16 people were recorded as "yeses" in a sample of 40 people, the sample proportion of "yeses" would still be .4. Using the table of 90% box plots for samples of size 40 indicates that .40 is a likely sample proportion for populations from 30% to 50%. The results should be recorded as 40% $\pm$10% in nine out of 10 cases.

A major portion of the evaluation of this concept should be the actual design and evaluation of a survey. The focus should be on their environment and answering questions that are of interest to them. Every effort should be made to ensure a truly representative sample and to be aware of any bias that might creep into their results.

PROCESS/PROBLEM SOLVING CONTEXT

Activity 1: Introduce the idea of surveys and their terminology by asking a question of everyone in the class. The question should be of the "yes-no" variety.

a) Do you like watching NFL football on TV?
b) Did you do more than one hour of homework last night?
c) Does your family have a CD player?

Calculate the percentage of yeses for the class. Now randomly select five students and get their answers to the same question and calculate the mean for this sample of five. Compare the population and sample means. Repeat with several other samples.

Activity 2: Have students collect examples of surveys that are discussed in the paper or magazines. Note which ones are "yes-no" surveys and the results of the survey, especially the confidence interval.

Activity 3: Have a class cooperatively construct a 90% box plot with sample sizes of 20. Each group of two could work on a different population percentage of yeses using a random sample. These could be constructed using a variety of techniques, including the random number generator found on some calculators, random number tables, computer programs or even specially designed spinners. A spinner for a 40%-yes population, for example, could feature a circle divided into 10 "pieces" where 4 "pieces" are labelled "yes" and six "pieces" are labelled "no".

Activity 4: Ask students to construct a 95% box plot for samples of 20 and the population percentage of yeses as 40%. Compare with the 90% box plot. The simulations done previously can still be used. In simulations using 1000 trials, approximately 50 trials will be labelled as unlikely and the "middle" 950 as likely. To be 95% confident of the answer the range of possible values must be widened.

Other Activities: For a collection of activities, ideas and teaching strategies see the book Exploring Surveys and Information from Samples.[1]

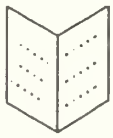
TECHNOLOGY INTEGRATION

As noted above, there are many opportunities to use computers and calculators in this unit. Some spreadsheets even have a random number generator. These can be used to do simulations of various populations.

1. Landwehr, James M. et al. Exploring Surveys and Information from Samples. Dale Seymour Publications, Palo Alto, CA, 1987.

RESOURCE CORRELATION

MMW4: pp. 272–275; 285–290
MM4: pp. 264–280



TRIGONOMETRY
SUGGESTED PROGRAM EMPHASIS – 14 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that angles can be drawn on a coordinate plane and that their trigonometric ratios can be determined with respect to the coordinates of points on the plane.	Students will be expected to recognize and sketch positive and negative angles in standard position on a coordinate plane. Students will be expected to determine the reference angle for an angle drawn in standard position on a coordinate plane. Students will be expected to express the trigonometric ratios for any angle, θ, in standard position on a coordinate plane in terms of x, y and r, where r is the length of the terminal side and (x, y) the coordinates of the end point. Students will be expected to determine the sine, cosine and tangent ratios for any angle. Students will be expected to determine any two values of x, y, r and θ given the other two. Students will be expected to use a graphing calculator or computer to investigate the graph of the sine curve.	Students will be expected to define and determine principle and coterminal angles on a coordinate plane. Students will be expected to use a calculator to determine the sine, cosine and tangent ratios of any angle. Students will be expected to use a graphing calculator or computer to investigate the effects of the parameters a and b on a function $y = a \sin(bx)$.
Students will be expected to demonstrate an understanding that the methods used in the solution of right triangles can be used to develop laws for use in the solution of oblique triangles.	Students will be expected to find the measures of sides and angles in diagrams involving multiple right triangles in two or three dimensions. Students will be expected to find the measures of unknown sides and angles in oblique triangles by applying the cosine law. Students are expected to find the measures of unknown sides and angles in oblique triangles by applying the cosine law.	Students will be expected to solve problems involving multiple right triangles in two or three dimensions. Students will be expected to illustrate the sine law using examples. Students will be expected to solve problems using the sine law. Students will be expected to illustrate the ambiguous case of the sine law. Students will be expected to illustrate the cosine law using examples. Students will be expected to solve problems using the cosine law. Students will be expected to recognize that the cosine law is the generalization of the Pythagorean Theorem.

CONCEPT 1

Students will be expected to demonstrate an understanding that angles can be drawn on a coordinate plane and that their trigonometric ratios can be determined with respect to the coordinates of points on the plane.

Students will be expected to:

1.1 recognize and sketch positive and negative angles in standard position on a coordinate plane
  1.1.1 define and determine principle and coterminal angles on a coordinate plane
1.2 determine the reference angle for an angle drawn in standard position on a coordinate plane.

COMMENTS

In Mathematics 23, students studied the three primary trigonometric ratios in relationship to the solution of right triangle problems. The intent in Mathematics 33 is to extend the study of trigonometry to oblique triangles.

Mathematics 33 students usually have difficulty dealing with the abstract. An inductive approach, therefore, is recommended before moving to the abstract. Connecting trigonometry to the coordinate system is difficult even with strong math students.

Emphasize that reference angles are positive acute angles formed by the terminal arm of an angle and the x-axis. Reference angles were required to find trig ratios for angles when tables were used instead of the technology we have today,

Note: The study is restricted to the primary trigonometric ratios so that students gain a complete understanding of sine, cosine and tangent. As students have calculators to find trigonometric ratios, introduction of the reciprocal ratios will not be necessary. In Mathematics 30 the inverses of sine, cosine and tangent functions are introduced.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have students complete a chart like the one following to see that the trig functions do exist for angles other than the acute angles of a right triangle. Students should use their calculators to find these ratios. Remind students that their calculators must be in degree mode.

Angle A	sin (A)	cos (A)	tan (A)
30°			
150°			
210°			
330°			
390°			
410°			
570°			
690°			
−30°			
−150°			
−330°			
90°			
180°			
270°			
360°			
1000°			
−1000°			
etc.			

Let students choose some angles themselves. Investigate ways of
a. drawing angles such as these
b. defining trig ratios for these angles
c. determining why some angles have the same trig ratios.

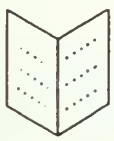
TECHNOLOGY INTEGRATION

Video:
Trigonometric Functions I (TV Ontario): #1, Trigonometric Ratios (review); #2, Solving Right Triangles (review); #3, Angles on the Plane.

These videos are best used for review rather than to introduce new topics because they move too fast for students to grasp the concept on their initial presentation.

RESOURCE CORRELATION

MMW4: pp. 184–186
MM4: pp. 194–196



CONCEPT 1 (continued)

Students will be expected to demonstrate an understanding that angles can be drawn on a coordinate plane and that their trigonometric ratios can be determined with respect to the coordinate of points on the plane.

Students will be expected to:

1.3 express the trigonometric ratios for any angle, θ, in standard position on a coordinate plane in terms of x, y and r, where r is the length of the terminal side and (x,y) the coordinates of the end point

1.4 determine the sine, cosine and tangent ratios for any angle

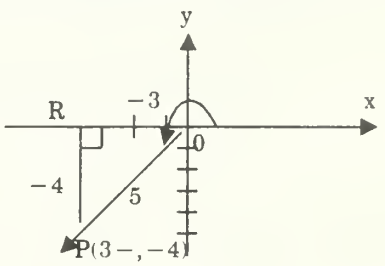
1.4.1 use a calculator to determine the sine, cosine and tangent ratios of any angle

1.5 determine any two values of x, y, r and θ given the other two.

COMMENTS

Mathematics 23 students have had no experience with trigonometry in the coordinate system. A review of the definitions for the sine, cosine and tangent ratios for acute angles in right triangles is recommended. From these definitions, the definitions for the trig functions for angles on a coordinate plane can be developed as done in MM4 and MMW4. Note the use of reference triangles in MM4, rather than just using the definitions of the trig functions for angles in a coordinate plane. Using a reference triangle helps tie the right triangle definitions of trig ratios to those for any angle in the coordinate system. For example, an angle drawn in standard position has P(−3, −4) on its terminal arm. Draw the angle and set up a reference triangle by drawing a vertical line segment from the point (−3, −4) to the x-axis.

ΔPOR is the reference triangle. Use <POR for the reference angle to determine the trig ratios for θ.



$r^2 = x^2 + y^2$
$r^2 = (-3)^2 + (-4)^2$
$r^2 = 9 + 16$
$r^2 = 25$
$r = 5, r > 0$

The reference triangle should have sides labelled −3, −4, and the hypotenuse can be calculated, using Pythagorean's Theorem, as 5. Using the reference angle in the triangle, the trig ratios can be determined by using the sides of this reference triangle.

$\sin(\theta) = -4/5$ [opposite side/hypotenuse]
$\cos(\theta) = -3/5$ [adjacent side/hypotenuse]
$\tan(\theta) = -4/-3$ or 4/3 [opposite side/adjacent side]

It is important that students be given the opportunity to make their own generalizations in terms of when the sine of an angle is positive or negative. Understanding what sign (positive or negative) the trigonometric ratio of a particular angle is important particularly when students examine the reasonableness of the result provided by a calculator. Hence, memorization of the "CAST" rule is not important, but understanding the relationships within the rule IS important.

Note: This beginning work in relating trigonometry as a circular function is important for students transferring to Mathematics 30.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Have students sketch several angles in standard position on a coordinate plane and determine the trigonometric ratios. Once this has been completed, have students examine the trigonometric ratios they have found to determine a relationship between the quadrant that the terminal arm of the angle lies in and the sign of the trigonometric ratio. This opportunity can lead to students examining the "circular" nature of trigonometry. Students will probably notice that the trigonometric ratios for angles in standard position are the same but that the signs differ – or that the sine ratio for some angle is the same as the cosine ratio for another angle. Examining these relationships will help students determine whether or not the answer provided by a calculator is reasonable.

Activity 2: To have students determine any two values of x, y, r and θ given the other two, use a table similar to the one below.

Emphasize that in finding θ given r and either x or y, two answers can be determined for θ and the calculator will give only one answer. To find the value of θ, use the reference angle in the reference triangle and then find the corresponding values of θ measured from the positive x-axis.

x	y	r	θ
3	4		
−3	4		
		5	Note: 2 answers
		2	45°
		1	−150°

Problem: Explain why the following statements are true.
$-1 \leq \sin\theta \leq -1 \leq$ and $1 \cos\theta \leq 1$

(Thanks to M. Ray, J.A. Williams High School)

Have students suggest several ways to solve this problem. For example:

1. Use a graphing calculator.
2. Find values for θ from the trig function.
3. Use the definitions of the trig functions and the fact that r is always greater than x or y.

Note: This problem could also be discussed when investigating the graphs of the sine and cosine functions.

TECHNOLOGY INTEGRATION

Note: Calculators must be in degree mode.

See MM4: p. 460–461 Trigonometry on a Spreadsheet.

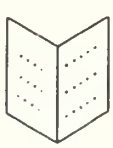
EXTENSION

1. Introduce polar coordinates (r, θ), as another way to describe angles in a coordinate plane and the point (x, y) on the terminal arm of θ.

2. Trig functions defined in the unit circle can be investigated.

RESOURCE CORRELATION

MMW 4:	pp. 186–192
MM4:	pp. 186–191; 194–203



CONCEPT 1 (continued)

Students will be expected to demonstrate an understanding that angles can be drawn on a coordinate plane and that their trigonometric ratios can be determined with respect to the coordinate of points on the plane.

Students will be expected to:

1.6 use a graphing calculator or computer to investigate the graph of the sine curve

1.6.1 use a graphing calculator or computer to investigate the effects of the parameters a and be on a function $y = a \sin(bx)$.

COMMENTS

Since students from Mathematics 23 have had little experience graphing linear functions investigating the graphs of the sine curve should be done after the unit on Functions and Relations. Investigating the graphs of $y = \sin x$ and $y = a \sin(bx)$ is an extension of the concepts dealt with in this unit. Using computers or graphing calculators, investigate the graphs for repeating patterns, their domain and range, and intercepts.

TECHNOLOGY INTEGRATION

Note: On graphing calculators, the ranges for the x and y axes should be adjusted to accommodate at least one cycle of the sine and graph. Their normal mode is radian measure so changing to degree mode may be more meaningful since Mathematics 33 students have not worked with radian measure.

EXTENSION

1. The graphs of $y = \cos x$ and $y = a \cos(bx)$ can be investigated and compared to those for the sine curve. Discuss the amplitude and period of the sine and cosine graphs. Since transformations of functions is done in the Functions and Relations Unit the horizontal and/or vertical translation of these trig graphs can also be done using computers or graphing calculators.

   See MM4: p. 207, or MMW4: pp. 196–205.

2. Investigate the graph of $y = \tan x$.

3. The following problem could be solved using the graphing calculator instead of with an algebraic approach.

Have students enter $y = x^2 + 1$ and $y = \cos x$. Their point of intersection gives the value for x that satisfies the equation.

Problem: Find all values of x that satisfy $x^2 - \cos x + 1 = 0$.

Answer: $x = 0$.

$$x^2 - \cos x + 1 = 0$$
$$x^2 + 1 = \cos x$$

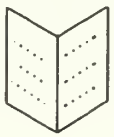
But $-1 \leq \cos \times \leq 1$

$\therefore \quad -1 \leq x^2 + 1 \leq 1$

$$-2 \leq x^2 \leq 0$$

Working with real solutions only, we have $x = 0$.

RESOURCE CORRELATION



MMW4: pp. 193–195
MM4: pp. 205–206

CONCEPT 2

Students will be expected to demonstrate an understanding that the methods used in the solution of right triangles can be used to develop laws for use in the solution of oblique triangles.

Students will be expected to:

2.1 find the measures of sides and angles in diagrams involving multiple right triangles in two or three dimensions

2.1.1 solve problems involving multiple right triangles in two or three dimensions.

COMMENTS

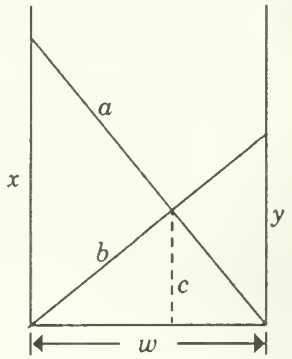
Solving right triangles in two and three dimensions is an extension of the Mathematics 23 program where students solve right triangles in two dimensions. A review of the definitions of the angle of elevation and the angle of depression will be necessary before solving the word problems. To introduce students to the necessity of indirect measurement and the use of trigonometry to solve these problems, have students do some outside activity such as the one suggested in the Process/Problem-Solving Context.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: Have students devise a method of measuring the height of a street light across the street from the school without crossing the street. They may use a tape measure and an instrument for measuring angles of elevation. Have them measure the angle of elevation of the light from a point outside the school, then move 10 feet closer and measure the angle of elevation from this new reference point. From these data, they should be able to go back to the classroom and determine the approximate height of the street light. Encourage students to try different methods of solving this problem, such as scale drawings and then an algebraic approach.

Problem: For Mathematics 33 students the following problem could be solved without using this algebraic approach. This approach may be too difficult but encourage students to find alternate ways to solve this problem. They could use a scale diagram, or build a scale model or use graph paper to draw the diagram. The algebraic solution could be attempted as an extension exercise for this objective.

Problem 2.1: Two ladders are propped across a narrow alley. The lengths of the ladders are a = 9m and b = 6m. The alley is w = 4m wide. Find the height, c, above the ground of the intersecting point of the two ladders.



Answer: c = 2.87m

One solution:

Using trigonometry:

$\cos\theta = {}^{w}/_{b} = {}^{4}/_{6}$

$\therefore\theta = 48.2°$

$\cos\theta = {}^{w}/_{a} = {}^{4}/_{9}$

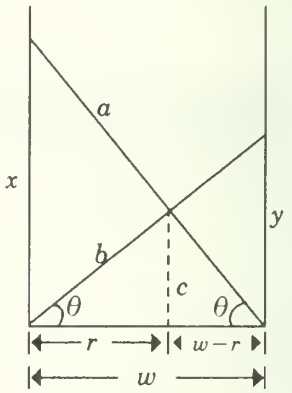
$\therefore\theta = 63.6°$

Using trigonometry, or Pythagorean Theorem:

$y = 4.47m$

$x = 8.05m$

By similar triangles:



$${}^{c}/_{4.47} = {}^{r}/_{4}$$

$$\therefore 4c = 4.47r$$

and

$${}^{c}/_{8.06} = {}^{4-r}/_{4}$$

$$\therefore 4c = 8.06(4-r)$$

$$= 32.24 - 8.06r$$

$$\therefore 4.47r = 32.24 - 8.06r$$

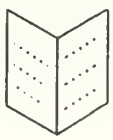
$$r = 2.57$$

Substituting,

$$4c = 4.47(2.57)$$

$$c = 2.87m.$$

RESOURCE CORRELATION



MMW 4: pp. 208–212

MM4: pp. 188–193; 212–221

CONCEPT 2 (continued)

Students will be expected to demonstrate an understanding that the methods used in the solution of right triangles can be used to develop laws for use in the solution of oblique triangles.

Students will be expected to:

2.2 find the measures of unknown sides and angles in oblique triangles by applying the sine law
- 2.2.1 illustrate the sine law using examples
- 2.2.2 solve problems by using the sine law
- 2.2.3 illustrate the ambiguous case of the sine law.

COMMENTS

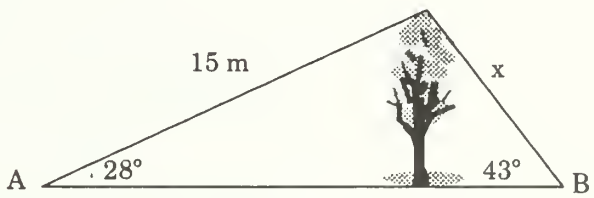
In introducing the Sine Law, it is important for students to see that it is developed by comparing triangles but it is not necessary to have students memorize the development or "proof."

The ambiguous case need only be illustrated. Illustrate this through the use of examples such as the ones in the Process/Problem-Solving Context. Solving of the two triangles is not required.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1: This problem could be used to introduce the development of the Sine Law through comparison of triangles.

From a point A, the angle of elevation of the top of a tree is 28°. From another point B, on the other side of the tree, the angle of elevation of the top of the tree is 43°. Two guy wires are attached from the top of the tree to the points A and B on the ground. If the guy wire to point A is 15 m long, how long is the guy wire to point B?



Solution: Let h be the height of the tree and x be the length of the guy wire to point B.

Then $\sin(28°) = h/15$ and $\sin(43°) = h/x$.

Thus $h = 15 \sin(28°)$ and $h = x \sin(43°)$.

Therefore $15 \sin(28°) = x \sin(43°)$.

From here they may solve for x without finding h.

The Sine Law may be demonstrated as well.

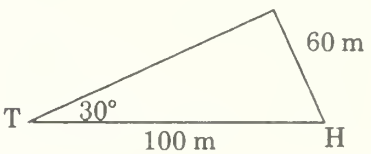
$$15/\sin(43°) = x/\sin(28°)$$

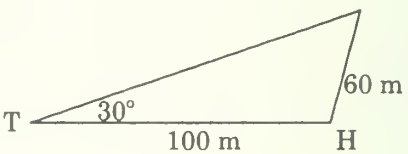
Answer: The other guy wire is 10.3 m long.

Activity 1: To verify that the ratios stated in the Sine Law are equal describe a triangle by giving a measure for all three sides and angles. Have students find the value of a/sinA, b/sinB and c/sinC to two decimal places. Are they equal? One triangle could be as follows: Triangle ABC where $<A = 15°$, $<B = 30°$, and $<C = 135°$, $a = 7.32$, $b = 14.14$ and $c = 20$.

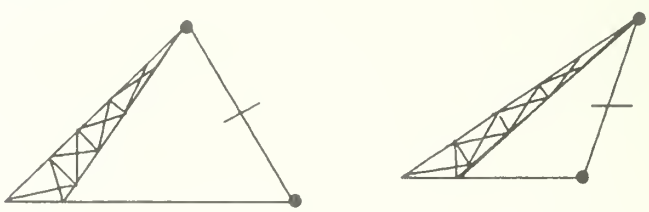
Activity 2: To illustrate the ambiguous case have students use protractors and a ruler or compass to draw triangles that will produce an ambiguous situation. For example, draw a triangle where $<A = 40°$, side $c = 6$ cm and side $a = 4$ cm.

Problem 2: On the sixth hole of a golf course the hole is 100 m from the tree. A golfer slices the ball 30° off the direct line to the hole. The ball lands 60 m from the hole. Draw diagrams to illustrate the two situations that could occur. The ball could land short of the hole or the ball could land past the hold.





Problem 3: To illustrate the ambiguous case a crane has a steel girder that must be placed in position on a building. Show the two positions that the steel girder can be placed by this crane. See diagram.



Problem 4: In a 5-12-13 right triangle, what is the length of the bisector of the larger acute angle?

Answer: $5\sqrt{13}/3 = 6.01$

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TECHNOLOGY INTEGRATION

See MMW4: p. 160, "Solving Triangles with a Computer"
See M11: pp. 427–429, "Computer Power"
See HM11: p. 314, "Computer Application – Law of Sines"

Video:
Trigonometric Functions I (TV Ontario): No. 4, The Sine Law; No. 6, Applications of the Sine Law and the Cosine Law.

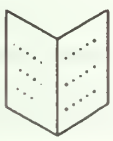
EXTENSION

Solve ambiguous cases.

Problem: In problem 2 have students find the distance that the golfer hit the ball on his first shot in the two situations. This could be determined from a scale drawing, or using an algebraic approach. The distances are 53.4 m and 119.8 m.

RESOURCE CORRELATION

MMW4:	pp. 216–219; 223–230
MM4:	pp. 222–226; 231–233



CONCEPT 2 (continued)

Students will be expected to demonstrate an understanding that the methods used in the solution of right triangles can be used to develop laws for use in the solution of oblique triangles.

Students will be expected to:

2.3 find the measures of unknown sides and angles in oblique triangles by applying the cosine law
- 2.3.1 verify the cosine law
- 2.3.2 solve problems by using the cosine law
- 2.3.3 recognize that the cosine law is the generalization of the Pythagorean Theorem.

COMMENTS

As with the sine law, the cosine law need not be proven. The second formula for the cosine law may be given for solving for an angle in a triangle, namely,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

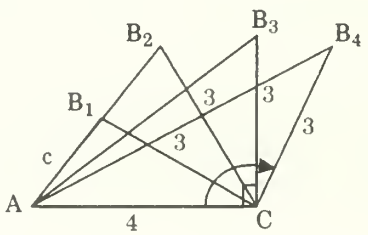
In solving for the largest angle of an oblique triangle, the cosine law will give the obtuse angle, if one exists, whereas the sine law will give an acute angle. Caution students about this.

The formulas must be memorized.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: To verify that the cosine law is true have students compare a^2 to the value of $b^2 + c^2 - 2bc\cos(A)$ and so on for the triangle where $<A = 15$, $<B = 30$ and $<C = 135$, side $a = 7.32$, side $b = 14.14$ and $c = 20$. Use this triangle to verify the second formula for the cosine law as well.

Activity 2: To compare the cosine law to the Pythagorean Theorem, have students complete the following chart. Use a triangle ABC where side a and side b remain constant and angle C increases from an acute angle to a right angle and then to an obtuse angle.



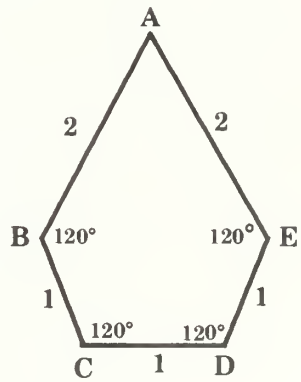
a	b	$<C$	c	c^2	a^2+b^2	compare c^2 to a^2+b^2
3	4	30°				
3	4	45°				
3	4	60°				
3	4	80°				
3	4	90°				
3	4	120°				
etc.						

Problem 1: Solve triangle ABC if $<B=22.3°$, side $a=5$ and side $c=4$. (Beware $<A$ is an obtuse angle.)

Answer: $b=2$, $<C=49.3°$ and $<A=108.3°$.

Note: If students use the sine law to find angle A, the calculator gives an acute angle and the measures of the remaining parts of the triangle lead to an inconsistent solution. That is, the largest angle in the triangle should be across from the longest side and the smallest angle should be across from the shortest side of the triangle. If the student uses the cosine law to find the largest angle, then the obtuse angle is given by the calculator.

Problem 2: How long is AC?



Answer: $\sqrt{7} \doteq 2.6$

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TECHNOLOGY INTEGRATION

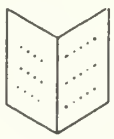
Bracket keys on the calculator are useful in finding the cosine of an angle using the second formula for the cosine law. These formulas may be programmed into a graphing calculator so memorization is not necessary.

See M11: pp. 427 – 429, "Computer Power"
See MPP11: p. 269, "Computer Insight: Classifying Triangles"

Video:
Trigonometric Functions I (TV Ontario): No. 5, The Cosine Law; No. 6, Applications of the Sine Law and Cosine Law.

RESOURCE CORRELATION

MMW 4:	pp. 219–230
MM4:	pp. 227–233



POLYNOMIALS AND RATIONAL EXPRESSIONS
SUGGESTED PROGRAM EMPHASIS – 8 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that rational expressions are fractions in which the numerators and denominators are polynomials and that operations can be performed on such expressions.	Students will be expected to determine the non-permissible values in rational expressions in one variable. Students will be expected to simplify rational expressions by factoring. Students will be expected to perform the operations of multiplication and division on rational expressions. Students will be expected to perform the operations of addition and subtraction on rational expressions. Students will be expected to solve equations involving rational expressions.	Students will be expected to use a calculator or computer to graph rational functions to determine non-permissible values. Students will be expected to solve problems involving equations containing rational expressions.

CONCEPT 1

Students will be expected to demonstrate an understanding that rational expressions are fractions in which the numerators and denominators are polynomials and that operations can be performed on such expressions.

Students will be expected to:

1.1 determine the non-permissible values in rational expressions in one variable
  1.1.1 use a calculator or computer to graph rational functions to determine non-permissible values
1.2 simplify rational expressions by factoring.

COMMENTS

Students have not done any previous work with rational expressions. Review what a polynomial is with its restrictions on negative or rational exponents (x^{-2} and $x^{\frac{1}{2}}$ are not polynomials). In Mathematics 23 students have studied the following types of factoring: common factors; difference of squares; ax^2+bx+c where a, b, c are integers. Emphasize all types so that the simplification of rational expressions becomes a factoring exercise instead of just cancelling terms. The sum and difference of cubes should not be included in the study of rational expressions.

The following question illustrates the need for factoring:

$$\frac{2x^2-8}{x^2-4x+4} \div \frac{3x+6}{x^2+2x}$$

a. Without simplifying, how does the value of the expression change if x increases in value from 1 to 3?
Answer: -2 to 10

b. How does the value of the <u>simplified</u> expression change if x increases in value from 1 to 3? Students must factor to avoid cancelling the x^2, the 2s and the 4s which some may be tempted to do. The answers must be the same so the skill of factoring is necessary.

The use of a factor of -1 should also be reviewed, i.e., that $\frac{(a-b)}{(b-a)} = -1$.

Which of the following are equivalent to $\frac{y-x}{x+y}$:

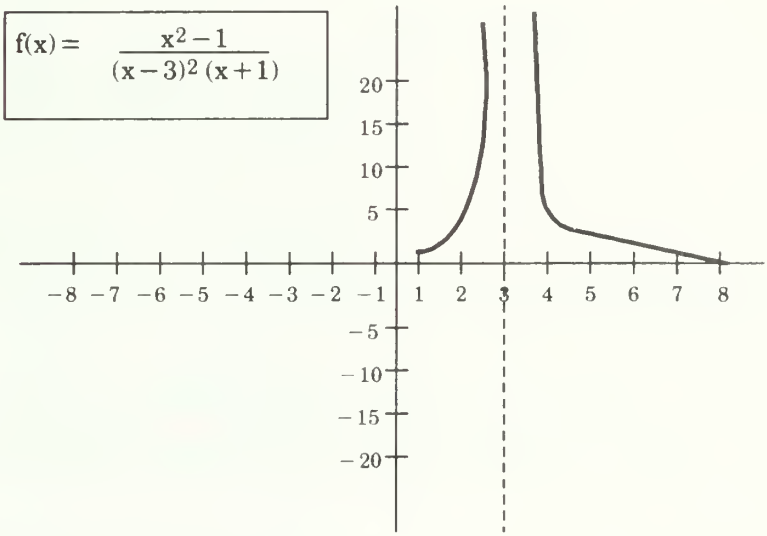
a. $\frac{-x-y}{x+y}$ b. $\frac{-y-x}{x+y}$ c. $-\frac{-(y-x)}{x+y}$ d. $\frac{y-x}{y+x}$

Non-permissible values: A visual presentation with a computer or graphing calculator to determine non-permissible values adds an increased understanding of what it means for the rational expression to be invalid at a certain point or points. The denominator of a fraction cannot be equal to zero because it is undefined.

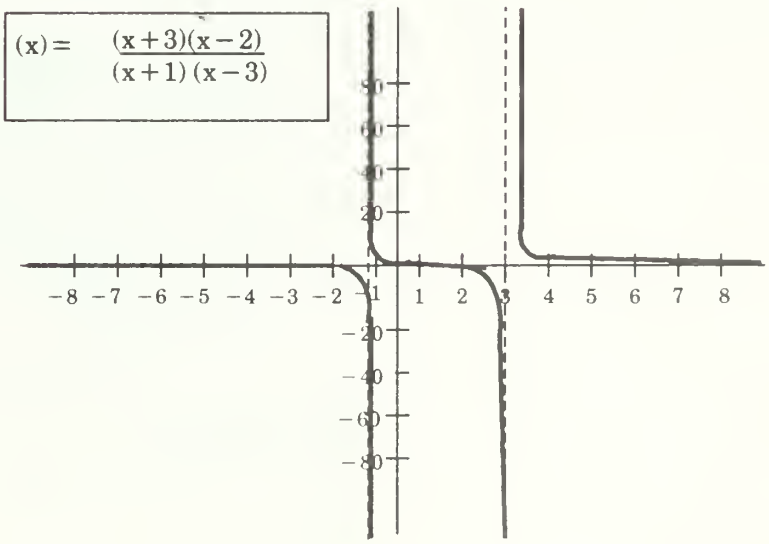
Note: When graphing rational expressions where factors do not divide evenly, the computer or calculator shows non-permissible values with an asymptote. But, when a factor does divide, the graph illustrates the answer only and the non-permissible value from that factor disappears.

The following graphs demonstrate this problem.

$$f(x) = \frac{x^2 - 1}{(x-3)^2 (x+1)}$$



$$(x) = \frac{(x+3)(x-2)}{(x+1)(x-3)}$$



Caution: When a factor in the denominator does divide out, a "pinpoint hole" is created and technology seems limited in illustrating this. In graphing rational equations, it is recommended to graph both sides of the equation separately.
e.g., $\frac{x}{x+1} = \frac{x^2}{x^2+x}$

This will illustrate the "hole" mentioned.

Students may not have solved equations with a variable in the denominator. Non-permissible values from this type of question will have to be ones where a quadratic results that is easily factorable by the methods already mentioned.

e.g., $x-2=\frac{8}{x}$, $x \neq 0$, $x=4$ and -2

Finding the solutions to this equation by letting the factors equal zero can lead into finding non-permissible values when binomials occur as factors in the denominator.

Note: Students should identify non-permissible values for rational expressions before simplifying
$\frac{x^2+x-2}{x^2+3x+2}$

$= \frac{(x+2)(x-1)}{(x+2)(x+1)}$ $x \neq -2, -1$

$= \frac{(x-1)}{(x+1)}$ $x \neq -2, -1.$

A real life example of non-permissible values can be seen in the following example.

Force of gravity $= \frac{(4.0\times10^{14})\,M}{r^2}$
where M = mass and r = distance from centre of each.

For an 85 kg astronaut on earth's surface (6.37×10^6 m from the earth's centre) the force of gravity is

$Fg = \frac{(4.0\times10^{14})(8.5\times10^{1})}{(6.37\times106)^2}$
$= 830$ N (newtons).

Find the force of gravity for the same astronaut (3.6×10^7) from the earth's centre.

Answer: 26 N (distance from centre of the earth can never be zero).

PROCESS/PROBLEM-SOLVING CONTEXT

Problem: For what replacement of x will $\frac{x-5}{x^3-16x}$ not have a reciprocal?

Answer: $x=5$

(Thanks to F. Heinen, Lethbridge Collegiate Institute.)

Problem: For how many values of x is $(x^2-9)(x-1)=(x^2-4x+3)(x+3)$?

Answer: all values

(Thanks to F. Heinen, Lethbridge Collegiate Institute.)

ELECTIVE

Find non-permissible values for complex fractions.

$$\frac{\frac{a}{1+a}+\frac{1-a}{a}}{\frac{a}{1+a}-\frac{1-a}{a}}$$

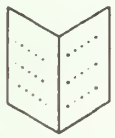
$a \neq 0, -1, \pm\frac{1}{\sqrt{2}}$

TECHNOLOGY INTEGRATION

Software:

- Master Grapher (Addison-Wesley), "Function Grapher" option.
- Computer Graphing Experiments (Addison-Wesley).
- Factoring Algebraic Expressions, (Mindscape) – review.

RESOURCE CORRELATION



MMW4: pp. 59–60
MM4: pp. 65–67

CONCEPT 1 (continued)

Students will be expected to demonstrate an understanding that rational expressions are fractions in which the numerators and denominators are polynomials and that operations can be performed on such expressions.

Students will be expected to:

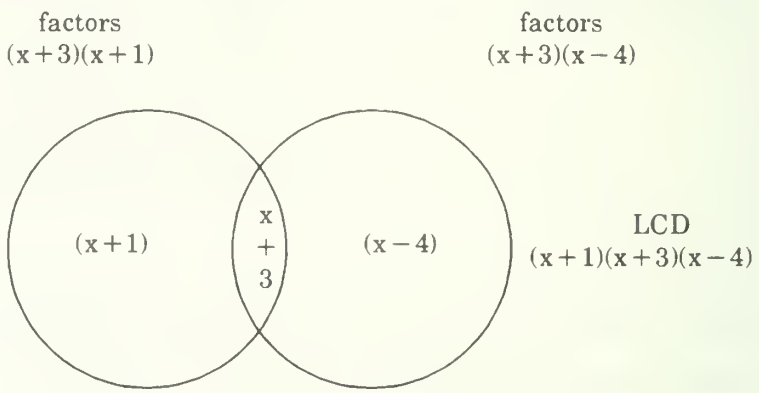
1.3 perform the operations of multiplication and division on rational expressions
1.4 perform the operations of addition and subtraction on rational expressions
1.5 solve equations involving rational expressions
  1.5.1 solve problems involving equations containing rational expressions.

COMMENTS

Simplifying rational expressions in 1.2 is extended to multiplication and division in 1.3. Note the limitation in 1.4 – students are only required to add or subtract with two rational expressions.

Finding a common denominator when adding and subtracting is often a problem. Illustrating this concept with overlapping circles may be an approach to try.

$$\frac{x+2}{x^2+4x+3} + \frac{x+1}{x^2-x-12}$$



In 1.5 you should limit rational expression equations to no more than a degree of one until students have studied the unit on solving quadratic equations. Students probably have not studied in Mathematics 23 equations with a variable in the denominator. Common denominators become significant again since clearing of fractions creates a workable equation.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 1.5.1: If the reciprocal of $x+1$ is $x-1$, then $x =$ ____.

Answer: $x = \pm\sqrt{2}$

Problem: The fraction $\frac{(5x-11)}{(2x^2+x-6)}$ was obtained by adding the two fractions $\frac{A}{(x+2)}$ and $\frac{B}{(2x-3)}$. Find the values of A and B.

Answer: $A = 3$ and $B = -1$

This problem also provides a review of solving a system of two equations in two unknowns.

Problem: To solve the equation $\frac{x-5}{x} = \frac{x-5}{x+1}$, Jake reasoned as follows:

"Since the numerators are equal, the denominators are equal. But this is impossible since x cannot equal x + 1. Therefore, there is no solution." Is Jake correct? If not, what is the error in his reasoning? What is the solution?
(Thanks to F. Heinen, Lethbridge Collegiate Institute.)

Problem: The area of a rectangle is represented by the expression $A = \frac{1}{(x-2)^2}$. Find the width of the rectangle if the length is represented by the expression $l = \frac{(3x-1)}{(2x^2+3x-14)}$.
(Thanks to M. Ray, J.A. Williams High School.)

Answer: $w = \frac{2x+7}{(x-2)(3x-1)}$

Problem: If $x+y=6$ evaluate:

$$\frac{x}{x-3} + \frac{y}{y-3}$$

Answer: 2

This problem emphasizes common denominator and some simple common factors.

Problem: If $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$

Express z in terms of x and y.

Answer: $z = \frac{xy}{y-x}$

Problem: The product of two numbers is four, and their sum is 12. Find the value of $\frac{1}{x}+\frac{1}{y}$

Answer: $\frac{1}{x}+\frac{1}{y}=\frac{(x+y)}{xy}=\frac{12}{4}=3$

Problems involving distance, speed and time apply the skills of rational expressions.

Problem: If a car travels 30 km/h slower, it will take 3 h more to travel 700 km. How fast is the car travelling now? (Make use of a chart.)

$$\frac{700}{s-30}-\frac{700}{s}=3$$

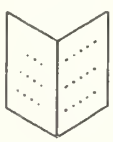
Answer: 100 km/h.

Problem: Find an expression in simplified form if you increase the difference between $\frac{1}{y+3}$ and $\frac{3y-1}{y^2-9}$ by $\frac{4}{3-y}$.

Answer: $\frac{-6y-14}{y^2-9}$

RESOURCE CORRELATION

MMW4: pp. 57–59; 62–64; 69–72
MM4: pp. 63–64; 68–76



FUNCTIONS AND RELATIONS
SUGGESTED PROGRAM EMPHASIS – 16 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that certain observed real-world phenomena are quantitatively related to each other and that these relations can be described graphically, with sets of ordered pairs, rules and equations.	Students will be expected to graph relations that describe physical phenomena or everyday situations.	Students will be expected to solve problems by graphing and interpreting the graphs that describe physical phenomena and everyday occurrences.
Students will be expected to demonstrate an understanding that for some relations, called functions, the value of the independent variable (domain) uniquely determines the value of the dependent variable (value of the function, range).	Students will be expected to represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and range, appropriately noting any restrictions. Students will be expected to interpolate and extrapolate from the graphs of simple functional relationships. Students will be expected to use functional notation and graphs to describe simple relationships and evaluate them by substituting appropriate replacements of the domain. Students will be expected to demonstrate an understanding that for some relations, called functions, the value of the independent variable (domain) uniquely determines the value of the dependent variable (value of the function, range).	Students will be expected to solve problems algebraically or by the use and interpretation of graphs that represent functions. Students will be expected to draw and analyze the graphs of functions using calculators or computers.
Students will be expected to demonstrate an understanding of how particular parameters can be used to effect translations, reflections or vertical stretchings of any function.	Students will be expected to describe the transformation effects on the graph of $y = f(x)$ of the parameters a and b in $y = f(x - a) + b$. Students will be expected to describe the transformation effects on the graph of $y = f(x)$ of the parameter c in $y = cf(x)$. Students will be expected to describe and sketch the graphs of $y = cf(x - a) + b$.	Students will be expected to investigate the effects of the parameters a and b on the graph of $y = f(x - a) + b$. Students will be expected to investigate the effects of the parameter c on the graph of $y = cf(x)$. Students will be expected to use calculators or computers to graph functions in the form $y = cf(x - a) + b$.

CONCEPT 1

Students will be expected to demonstrate an understanding that certain observed real-world phenomena are quantitatively related to each other and that these relations can be described graphically, with sets of ordered pairs, rules and equations.

Students will be expected to:

1.1 graph relations that describe physical phenomena or everyday situations

1.1.1 solve problems by graphing and interpreting the graphs that describe physical phenomena and everyday occurrences.

COMMENTS

This unit is an extension of the Relations unit from Mathematics 13. There are some excellent activities in that unit of the Mathematics 13 Teacher Resource Manual. Linear relations are an important topic discussed in Mathematics 23. The concept of correlation and the technique of scatter plots discussed in the Mathematics 33 Statistics unit would be an excellent introduction to this unit.

Students should be given experiences in a number of areas. Some examples of each area are given in the Process/Problem-Solving section.

1. "Thought" Experiments: These are situations where a graph is drawn of two variables that are related. No actual formula or numbers are necessarily involved.

2. Graph Interpretation: A graph of two related variables is given. Questions about the graph are asked. A formula used to produce the graph is not usually given.

3. Actual experiments are conducted with simple apparatus.

4. Tables of values are created from a calculable situation. In some of these questions it may be appropriate to write a formula that relates two variables.

5. Formulas are given that can be investigated.

6. Using the calculator as a "black box". One of the frequently used analogies of a function is that of a box that somehow transforms numbers into new numbers by some interior logic. The calculator does precisely that. The graphs of some functions like the "log", "sin", "n!" or "tan−1," can be graphed without fully understanding what these functions do. The fact that some input values give an error can lead to a domain and range discussion.

The concepts in the Functions and Relations unit are difficult to do in isolation. For example, it may be useful to introduce function notation early in the unit. Domain and range are also best introduced in the context of problems like the ones that follow.

PROCESS/PROBLEM-SOLVING CONTEXT

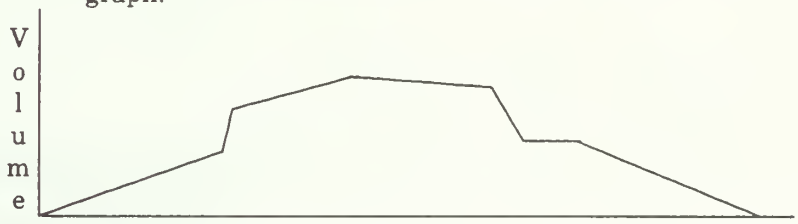
Activity: See the Comments section for a brief summary of each area described below.

1. Thought experiments

   a. Graph your height versus your age.
   b. Graph the temperature of a glass of water versus time after an ice cube is added.
   c. Given the profile of a roller-coaster ride, draw a graph of the speed of a car versus distance along the trace.[1]
   d. Given a map of a racetrack draw a graph of speed versus position on the track. The layout of a particular track is occasionally given in the newspaper. Alternatively, students could be asked to make their own design.

2. Interpretation of graphs

   a. A graph of speed versus time on a roller-coaster or a racetrack is given. The student attempts to draw a profile of a roller-coaster or racetrack.
   b. A graph of the volume of water in a bathtub is given. Students are asked to give a written explanation of the graph.



   Also see the Mathematics 20 Teacher Resource Manual, Functions and Relations unit.

3. Actual Experiments

   a. The length of a spring is graphed versus amount of mass added to the spring. This kind of material is typically available in a science department.
   b. The period of a pendulum is the amount of time it takes to complete one swing and return to its starting position. The period is graphed versus the length of the pendulum.

4. Table of values from calculable situations

   a. Telephone Pole Problem[2]

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1. NCTM Curriculum and Evaluation Standards. p. 83.
2. Problem Solving in Mathematics: Focus for the Future. p. 57

b. A ferris wheel has a radius of 10 m. It takes 40 seconds to complete one complete revolution. If a chair comes within 1 m of the ground, draw a graph of height versus time for two complete revolutions.

c. A farmer has 100 m of fencing. He wants to build a rectangular pen. Draw a graph of the width of the pen versus the area of the pen.

d. A phone call to Vancouver costs 82 cents for the first minute and 62 cents for each additional minute or portion of a minute. Draw a graph of cost versus time.

e. Triangles are built with matchsticks. Create a table of values and graph the number of matchsticks versus the number of triangles. Write a formula that relates the two variables.

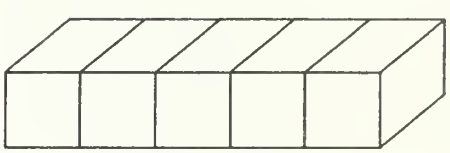
Answer:

t	m
1	3
2	5
3	7

$m(t) = 2t + 1$

f. Cubes are butted together. As a result some of the faces are not "hidden". Create a table of values, graph and formula that relates the number of cubes used and the number of faces.

Answer:



c	f
1	6
2	10
3	14
4	18

$f(c) = 4c + 2$

g. The number of games played in a sports league depends on the number of teams in the league. Assume that each team plays each other team twice. Draw a graph and write a formula that relates the number of games and the number of teams.

Answer:

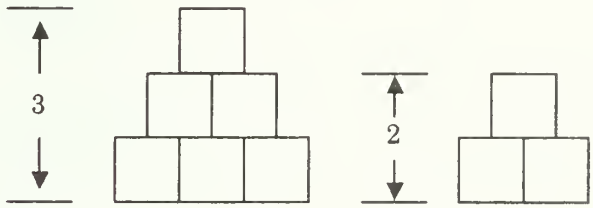
teams	games
2	2
3	6
4	12
5	20

The formula may be difficult for some to see. The number of games is the product of the number of teams and one less than the number of teams.

games = teams × (teams − 1)

$y(t) = t(t-1)$

h. Pyramids are built with blocks. The height of the pyramid is related to the number of blocks used. Graph and write a formula.



Answer:

blocks	height
1	1
2	3
3	6
4	10

The product of the number of blocks and one more than that is always twice the height.

$\text{height} = \frac{\text{blocks} \times (\text{blocks}+1)}{2}$

or

$h(b) = \frac{b(b+1)}{2}$

5. Formulas are investigated

For each formula below:
a. write in function notation
b. graph
c. discuss domain and range.

i. The area of a circle, A, is given by the formula $A = \pi r^2$.

ii. An object is thrown vertically. Its height, h, above the ground is given by the formula

$h = 22t - 5t^2 + 2$. (h in metres, t in seconds).

iii. The sum of the internal angles of an n-sided polygon is given by the formula

$S = 180(n-2)$.

iv. In a league with t teams, each team plays the other teams three times. The number of games, G, played is given by the formula

$$G = \frac{3t(t-1)}{2}.$$

6. The calculator as a "black box" activities

Make a table of values and graph using the following keys on the calculator: log, sin, n! and square root. The purpose of these is not to understand the mathematical significance of these keys but to illustrate that two numbers are "related" through the use of these functions.

Problem: Given the tables of values below, find the missing numbers and write a formula that "describes" the table.

a.

a	b
2	3
4	5
6	7
9	?

b.

r	m
1	2
2	5
3	8
4	?

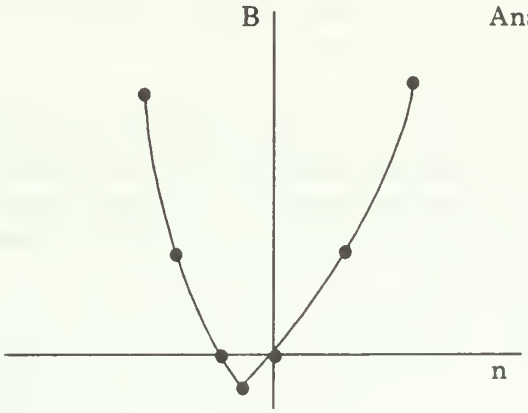
c.

h	k
2	5
3	10
4	17
5	?

Answer: a. 10, $b(a) = a + 1$
b. 11, $m(r) = 3r - 1$
c. 26, $k(h) = h^2 + 1$

Problem: Ask individual students to create a table of values based on a formula that they have in mind. The rest of the class attempts to "discover" the formula. Give them a goal that the question should be sufficiently difficult so that only 75% of the class will find the formula.

Problem: Give students a graph. Ask them to find a formula that describes the relationship. (They should probably build a table of values from the graph.)



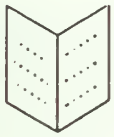
Answer:

n	B
−4	8
−3	3
−2	0
0	0
1	3
2	4

Focus on the positive "n" values. B is the product on "n" and two more than "n."

$B(n) = n(n+2)$

RESOURCE CORRELATION MMW4:
MM4: pp. 114–116



CONCEPT 2

Students will be expected to demonstrate an understanding that for some relations, called functions, the value of the independent variable (domain) uniquely determines the value of the dependent variable (value of the function, range).

Students will be expected to:

2.1 represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and range, appropriately noting any restrictions

2.2. interpolate and extrapolate from the graphs of simple functional relationships

2.3 use functional notation and graphs to describe simple relationships and evaluate them by substituting appropriate replacements of the domain

2.3.1 solve problems algebraically or by the use and interpretation of graphs that represent functions.

COMMENTS

Some of the objectives of this unit may have been introduced by doing the questions for the previous concept. This concept deals with variation. The study of variation should be limited to giving situations that are examples. The traditional study of variation is not the intent of this objective. This objective may be met without even mentioning the words direct, partial or inverse variation.

It is not always obvious which of two variables is dependent on the other. For example, if a graph is made of distance travelled in a car versus the amount of gas used, the independent variable can be identified by asking a question. The question to be answered is what makes more sense: a. the amount of gas depends on the distance travelled or b. the distance travelled depends on the amount of gas used.

Recall that Linear Relations is a major topic in Mathematics 23. Some students will remember certain techniques for writing the equations of lines. The primary method of "equation-writing" in this concept should not, however, involve algebraic methods. The equation should be written as a result of a relationship between two variables that the student could write in sentence form after studying the table of values. For example, using the table of values below, a student may see that the value in the "m" column is always obtained by multiplying the "n" value by two and increasing that by seven. This leads to the equation $m(n) = 2n + 7$.

n	2	3	4	5	6
m	11	13	15	17	19

The function that is written to describe the variation example should be used to make predictions for values of the independent variable that are not in the table of values (extrapolation and interpolation). Not all of the values of the independent variable will yield an "acceptable" value of the dependent variable when the function is used. For example, a formula for renting a car for one day is $Cost = .10 \times Xn + 30$ where n is the number of kilometres driven in one day. Evaluating the formula for $n = 5000$ may not be appropriate because that is an unusually large distance to drive in one day. The car company probably charges for every kilometre or portion thereof, meaning that evaluating for values of n like 500.2 would also be inappropriate.

The domain and range of these functions should be discussed. Domain and range should be written in sentence form rather than formal mathematical notation. For example, using the car rental formula from the previous paragraph might give the following domain and range.

Domain: The formula is limited to values of "n" that are whole numbers because the company charges for any portion of a kilometre. Driving the car for more than about 1000 km in one day would be unlikely.

Range: The lowest cost would be greater than $30 because that is the cost if the car is not driven. The maximum cost would be approximately $130.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity: For each situation below:

a. Identify the dependent and independent variables. Create a table of values.
b. Write a function that describes the table of values.
c. Graph the table of values. (Carefully consider whether the points should be joined together.)
d. Discuss the domain and range.

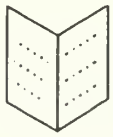
1. A car travels 7 km per litre of fuel used. The capacity of the tank is 70 L.

2. A car rental agency charges 12 cents per kilometre plus $20 per day.

3. The time required for money to double depends on the interest rate. Assume that money is compounded annually and that banks only offer whole number percents between 5% and 15%.

4. Square Deal Pizza makes square pizza. They charge 2.5 cents per square centimetre of pizza. They have only five sizes of pan: 15, 20, 25, 30 and 40 cm.

RESOURCE CORRELATION MMW4:
MM4: pp. 117–131



CONCEPT 2 (continued)

Students will be expected to demonstrate an understanding that for some relations, called functions, the value of the independent variable (domain) uniquely determines the value of the dependent variable (value of the function, range).

Students will be expected to:

2.4 illustrate and recognize different kinds of functions algebraically and graphically from the following list: linear functions, polynomial functions (including quadratic and cubic functions), reciprocal functions, absolute value functions and exponential functions

2.4.1 draw and analyze the graphs of functions using calculators or computers.

COMMENTS

The purpose of this concept is to introduce students to some of the ways that two variables can be related. They should be able to put relations into categories according to the formula used or the graph produced. The type of relations discussed should be limited to those listed in the objectives. Notice that there is no discussion of relations that are not functions.

Students will have used a number of different names for the dependent and independent variables in the previous concepts. In general, however, the letter "y" is used to represent any dependent variable and "x" represents any independent variable. Most graphing software will require that formulas be entered in this manner so students should be familiar with this idea.

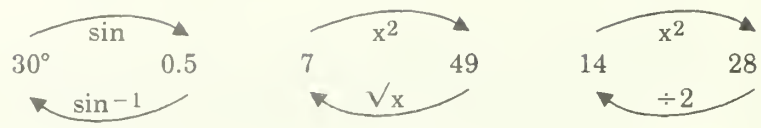
When given a formula, students should have some experience in building tables of values that can then be graphed. The vast **majority** of their time, however, should be spent using graphing software to produce graphs.

Domain and range questions should be asked only when the student has access to the graph or appropriate technology to produce the graph. The mathematical notation frequently used to indicate domain and range is considered an extension of the course. Students should be allowed to write the domain and range in sentence form even though it may lack the precision of the mathematical notation.

Students have encountered inverse functions previously:

a. $\sqrt{x}$ and x^2
b. sin and $\sin - 1$
c. multiplication and division by 2.

An inverse function "reverses" the effect of a function. For example:



See Activity 6 for further ideas.

The definition of the inverse of a function is based on a relationship between table of values. If the point (a,b) is in the table of values of a function then the point (b,a) is in the table of values for the inverse. Using function notation this can be stated as follows: If $f(a)=b$, then $f^{-1}(b)=a$. Therefore, if given the graph of a function a student could draw the graph of the inverse. (See Activity 5.)

In some simple situations a formula for the inverse can be found but this is not a primary intent of this objective.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Use the graphs on p. 191. Ask the students to cut out the graphs and place them in categories of their own choosing. The categories do not need to be "mathematical." The categories should each be given a name. The teacher can later give his/her categories.

Activity 2: Ask students to describe any type of relationship between two variables and then write an equation and draw a graph of that relationsnip. Does the resulting graph fit into one of the categories from Activity 1?

e.g., a. The value is the product of the x value and three less than the x value.

b. The y value is the difference between the square of x and 3 times x.

c. The y value is the difference between the square root of x and the cube root of x.

Activity 3: Ask students to describe a relationship and write an equation that fits into each one of the categories that the teacher defined from Activity 1. (Some will be very difficult.)

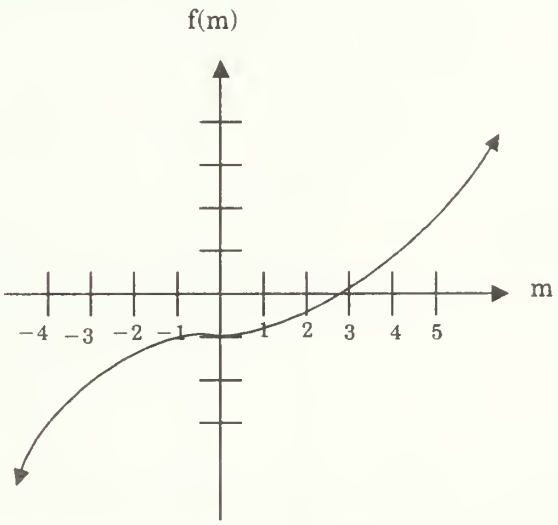
Activity 4: Graph and place into the teacher categories from question 1 each of the following equations.

a. $y = x$
b. $y = x^2$
c. $y = |x|$
d. $y = 1/x$
e. $y = 2^x$
f. $y = 3^x$

Activity 5: a. Use the graph below to find the following:

i. $f(-3)$ ii. $f(3)$ iii. $f(0)$ iv. $f(5)$

b. For what values of m is $(fm) > 0$?
c. Graph $f^{-1}(m)$
d. Find a. $f^{-1}(0)$, b. $f^{-1}(3)$



Activity 6: Draw any graphs that have domain and ranges expressed below. Show more than one answer for each.

a. The domain is any real number between -1 and 2 inclusive. The range is any real number between 2 and 5.

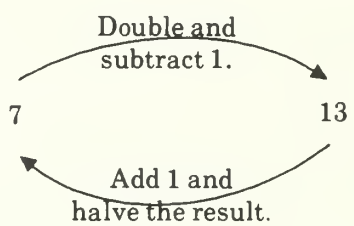
b. The domain is any integer between -2 and 3 inclusive while the range is any integer between -10 and 0 inclusive.

Activity 7: Fill in the table below.

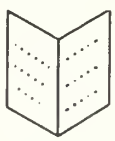
Description	Equation	Inverse Description	Inverse Equation
Double a number and subtract one.	$y=2x+1$	Add one and halve the result.	$y=(1+x)/2$
Square a number and add three.			
Add three to a number and cube the result			

Note: The inverse description would be what you need to do to a number to "reverse" the effect of the original function.

e.g.,

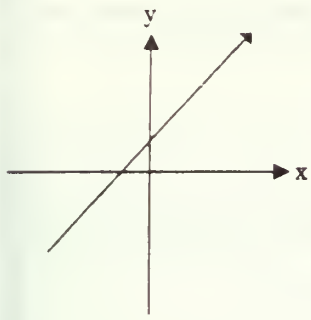


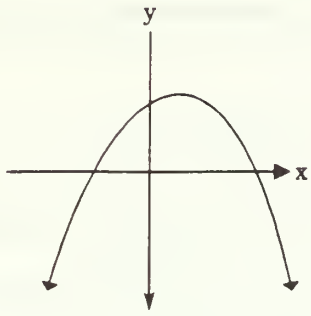
RESOURCE CORRELATION

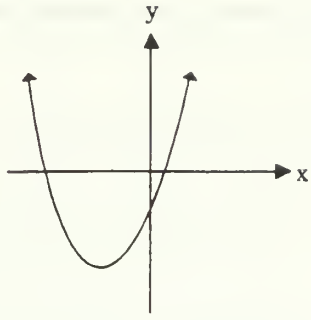


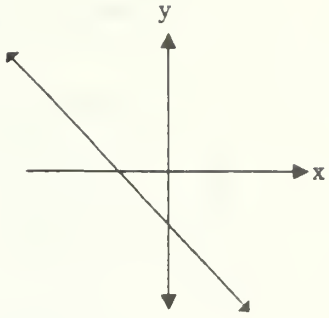
MMW 4:
MM4: pp. 135–137; 142–156

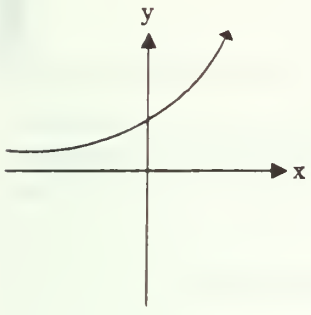
Activity 1 Worksheet

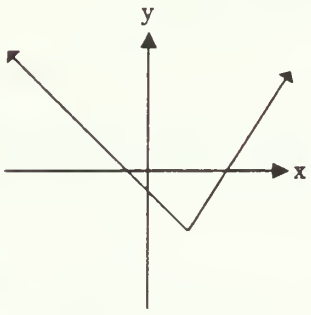


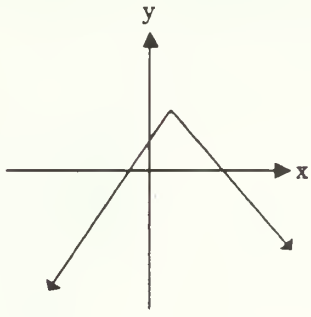


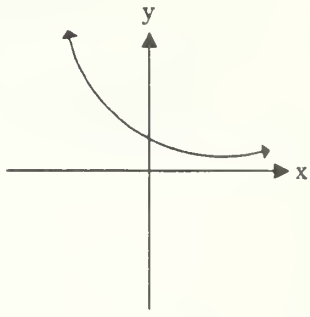


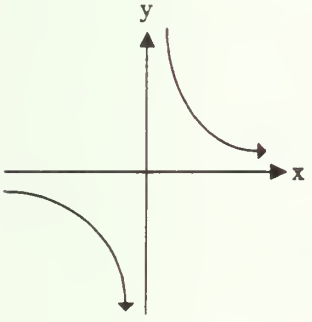


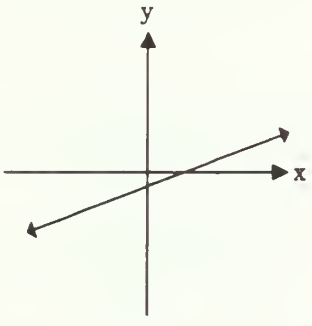


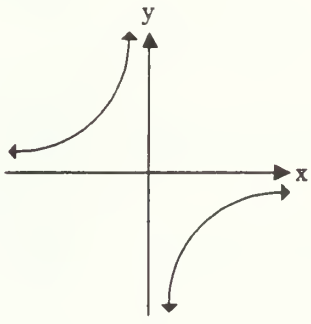


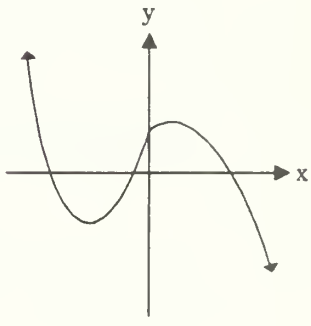


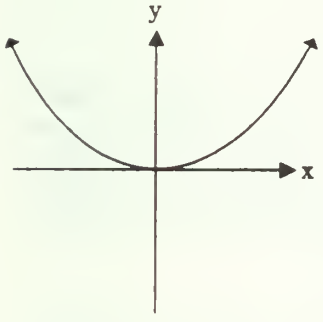


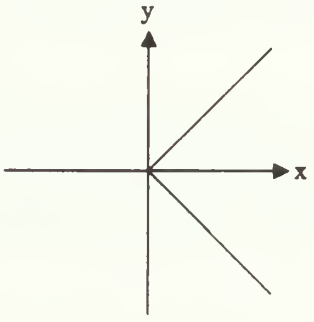


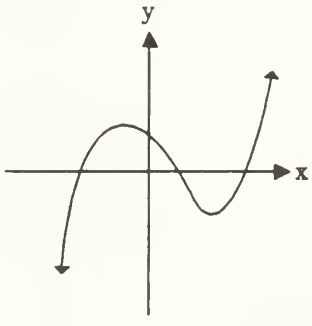


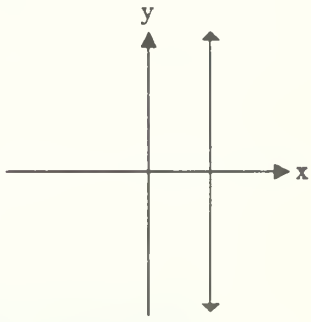


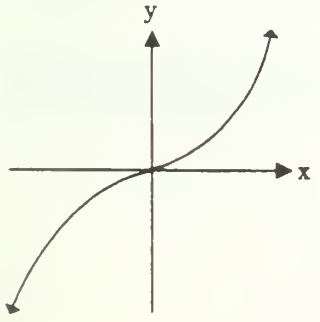


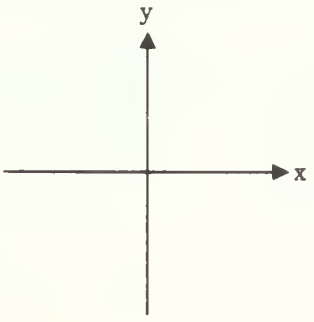












CONCEPT 3

Students will be expected to demonstrate an understanding of how particular parameters can be used to effect translations, reflections or vertical stretchings of any function.

Students will be expected to:

3.1 describe the transformation effects on the graph of $y = f(x)$ of the parameters a and b in $y = f(x - a) + b$

3.1.1 investigate the effects of the parameters a and b on the graph of $y = f(x - a) + b$

3.2 describe the transformation effects on the graph of $y = f(x)$ of the parameter c in $y = cf(x)$

3.2.1 investigate the effects of the parameter c on the graph of $y = cf(x)$

3.3 describe and sketch the graphs of $y = cf(x - a) + b$

3.3.1 use calculators or computers to graph functions in the form $y = cf(x - a) + b$.

COMMENTS

Once students are familiar with a few basic types of functions they can "create" new ones by transforming the originals. Different effects are achieved by doing one or more of the following:

1. Adding or subtracting a constant to the function
   e.g., $y = x$ becomes $y = x + 3$
   or $y = x^2$ becomes $y = x^2 - 5$.

2. Replace with $(x + a)$ or $(x - a)$
   e.g., $y = |x|$ becomes $y = |x + 4|$
   or $y = x^2 + 3$ becomes $y = (x - 5)^2 + 3$.

3. Multiply by a positive constant
   e.g., $y = x - 5$ becomes $y = 2x - 10$
   or $y = y^2 + 2x$ becomes $y = 3x^2 + 6x$.

4. Multiply by a negative constant
   e.g., $y = 3x - 1$ becomes $y = -6x + 2$.

Most of this concept should be done in a "self-discovery" mode. Students could be given an activity similar to Activity 1 and graphing technology to discover the effects of the above transformations. The effects of the transformations should be written in paragraph form. Different students may see the same transformations in slightly different ways.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 1: Graph each set of equations on the same set of axes.

Set 1→ $y = x^2$	$y = x^2 - 1$	$y = x^2 + 3$
Set 2→ $y = \|x\|$	$y = \|x\| - 1$	$y = \|x\| + 2$
Set 3→ $y = x^2 + 1$	$y = (x - 1)^2 + 1$	$y = (x + 2)^2 + 1$
Set 4→ $y = \|x\| - 2$	$y = \|x - 1\| - 2$	$y = \|x + 3\| - 2$
Set 5→ $y = x^2 + 2$	$y = 2x^2 + 4$	$y = 3x^2 + 6$
Set 6→ $y = x^2 - 2$	$y = -2x^2 + 4$	$y = -3x^2 + 6$

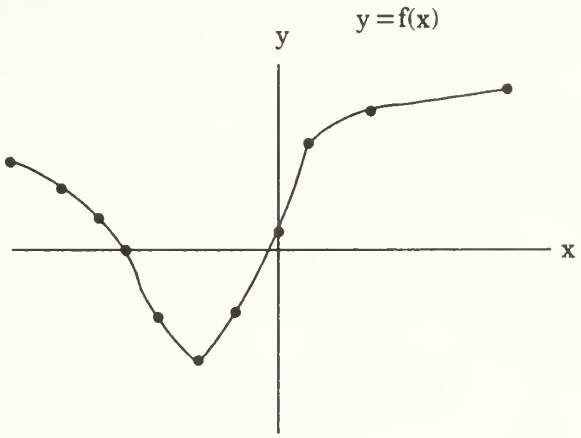
Answer the following questions for each set of equations.

1. What has been done mathematically to the first equation in each set of three equations?

2. What happens to the first graph as a result of the changes in the equation?

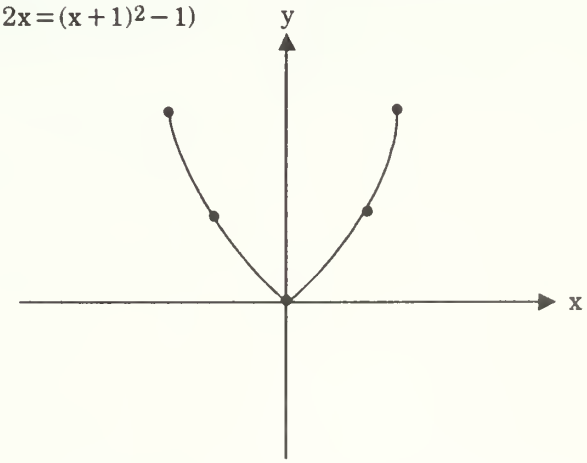
Problem: Given $y = f(x)$ below sketch the following graphs:

a. $y = f(x) - 1$ b. $y = 2f(x)$ c. $y = -f(x)$ d. $y = f(x-2)$
e. $y = 2f(x+1) + 2$

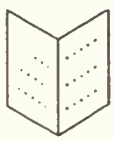


Problem: This is a graph of y = x2. Draw in a new x and y axis so that the graph becomes:

a. $y = (x+2)^2$ b. $y = x^2 + 2$ c. $y = (x-3)^2 - 1$
d. $y = x^2 + 2x$
(**Hint**: $x^2 + 2x = (x+1)^2 - 1$)



RESOURCE CORRELATION



MMW 4:
MM4: pp. 138–141; 157–158

QUADRATIC FUNCTIONS AND EQUATIONS
SUGGESTED PROGRAM EMPHASIS – 17 %

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding that any function that can be written in the form $y = ax^2 + bx + c$, $a \neq 0$, is a quadratic function and that individual quadratic functions have unique characteristics and graphs.	Students will be expected to sketch the graphs of quadratic functions written in standard form, $y = a(x - h)^2 + k$. Students will be expected to transform quadratic functions from the general form $y = ax^2 + bx + c$ to the standard form $y = a(x - h)^2 + k$ by completing the square.	Students will be expected to investigate the effects of the parameters a, h and k in $y = a(x - h)^2 + k$ using a graphing calculator or computer. Students will be expected to find the vertex, axis of symmetry, domain, range, maximum or minimum values, and y-intercepts of a quadratic function. Students will be expected to solve problems that involve quadratic functions, by analyzing the functions in graphic or equation form.
Students will be expected to demonstrate an understanding that finding the x-intercepts of the quadratic function $y = ax^2 + bx + c$ is equivalent to solving the quadratic equation $ax^2 + bx + c = 0$.	Students will be expected to use the quadratic formula to solve quadratic equations. Students will be expected to recognize and explain the relationship between the x-intercepts of the quadratic functions and the roots of their corresponding equations. Students will be expected to solve equations with radicals and rational expressions whose solutions are the roots of quadratic equations.	Students will be expected to determine the x-intercepts of the quadratic function. Students will be expected to solve problems that involve quadratic equations. Students will be expected to solve problems that involve equations with radicals and rational expressions whose solutions are the roots of quadratic equations.

CONCEPT 1

Students will be expected to demonstrate an understanding that any function that can be written in the form $y = ax^2 + bx + c$, $a \neq 0$, is a quadratic function and that individual quadratic functions have unique characteristics and graphs.

Students will be expected to:

1.1 sketch the graphs of quadratic functions written in standard form, $y = a(x - h)^2 + k$

- 1.1.1 investigate the effects of the parameters a, h and k in $y = a(x - h)^2 + k$ using a graphing calculator or computer

1.2 transform quadratic functions from the general form $y = ax^2 + bx + c$ to the standard form $y = a(x - h)^2 + k$ by completing the square

- 1.2.1 find the vertex, axis of symmetry, domain, range, maximum or minimum values, and y-intercepts of a quadratic function
- 1.2.2 solve problems that involve quadratic functions, by analyzing the functions in graphic or equation form.

COMMENTS

If students have done the "Functions and Relations" unit, expectations 1.1 and 1.1.1 will be a review for many of them (1.1 and 1.1.1 helps students connect the two units). If the "Functions and Relations" unit has not been completed before this unit, you may want to consider doing it first the next time.

As has been stated elsewhere in teacher resource manuals, the order shown is not necessarily the order in which the course must be taught. A case in point is to consider expectation 1.2.1 after 1.1.1. In so doing, students have the opportunity to do some investigation of these properties in relation to the values of the parameters a, h and k.

To put expectation 1.2 in context, here's an analogy that might help students. Consider the exercise of shopping at a store, purchasing an item and having it delivered in a box. The item must now be assembled before it can be used. The general form of the quadratic function, $y = ax^2 + bx + c$, is the "boxed" form and the standard form, $y = a(x - h)^2 + k$ is the "useful" form. This may help students understand the relevance of algebraic techniques like "completing the square." Expectation 1.2 also helps students link the graphic representation of the quadratic function and the standard form of the equation. This development then leads to an increased acceptance of the quadratic formula (expectation 2.1).

With reference to expectation 1.2.1, these properties should be defined, well explained and illustrated with respect to the graph of a parabola. It is important that students understand them well if they are going to have success in this unit. This section can lend itself nicely to a student investigation such as the ones that follow. The students should have some familiarity with many of the translations discussed in this unit as a result of their study of relations and functions.

The applications (word problems) studied in Mathematics 33 should be reasonably straightforward. The problems below are approaching the higher end of difficulty. It is important for students to sense that their work is useful to some degree, but not at the risk of losing what has been accomplished by studying problems that are too abstract or too difficult for this level.

PROCESS/PROBLEM-SOLVING CONTEXT

A great way to introduce this section is by using a problem with which at least some of the students could identify. This may help students understand why this section is studied. Below are a couple of problems to consider.

Problem: Harvey Haywire is going to build a fence around a pasture which he is readying for some stock. The supply store is offering an unbelievable discount on the material that he needs but only has 300 m left in stock. Harvey plans to build a rectangular pasture with another fence in the interior running parallel to two of the sides. What are the dimensions that he should use to enclose the largest possible area?

Answer: Two sides of 75 m and three sides of 50 m.

If that one is not too attractive, try this one.

Problem: A travel group at school is organizing a trip and has a quote of $2000 per person if 100 or fewer people go, but is prepared to offer a discount of $16 per person for each person over 100 who joins the tour. At what number of travellers will the travel agency realize the maximum gross income?

Answer: 112.5 people (112 or 113 will be more realistic)!

At this point students may wish to try the problem using a graphing approach, a trial-and-error approach or by using a spreadsheet. All of these should be welcomed, but suggest that there is an algebraic solution that will be discussed after they have studied the material. If you forget to come back to the problem the students will usually remind you.

Activity 1.2.1: Once the students are familiar with the definitions of the parameter mentioned above, have them prepare a chart with headings like the one below:

Equation | Up/down | Wide/narrow | Vertex | Eq'n of Axis of Sym.| Range | Max./min.

Suggest to the students that they should graph equations from each of the following categories:

$$y = ax^2,\ y = ax^2 + k,\ y = a(x - h)^2 \text{ and finally, } y = a(x - h)^2 + k.$$

To make the investigation easier and more attractive, they should use computers and graphing calculators to draw the graphs. Students will investigate each graph drawn and make the appropriate observation under each heading. They should be forming their own conclusions as they go which should, in turn, be guiding their inquiry. This activity also corresponds very well with some of the discussion relating to expectation 1.1.1. As well it begins to emphasize the process of scientific inquiry. As was suggested in the preceding section, having the students write their own conclusions is a good way to have the students summarize their results.

Activity 1.1.1: Given:

A	B
$y=\frac{3}{5}x^2$	$y=7x^2$
$y=-9x^2$	$y=-9(x-2)^2$
$y=\frac{1}{2}x^2$	$y=\frac{1}{2}(x+4)^2+5$

Have students describe, in their own words, the changes that occur in the graph of the function as it moves from position A to position B.

(Thanks to S. Kizior, Westwood Community High School.)

The point to emphasize here is the degree to which the inquiry is student initiated but teacher guided. Students should hypothesize about to the effect a particular value will have on the graph and then use the technology available to draw the graph, which will then lead them to further inquiry. Exercises like this lend themselves to cooperative learning strategies and also to class presentations.

Activity 1.2: Direct the students to write some equations in the form $y=a(x-h)^2+k$ and then expand them to the form $y=ax^2+bx+c$. Next, graph both the equations and compare their results. Ask them why they are the same. This provides an introduction, along with a story like the one mentioned in the "Comments" section, to the technique of completing the square.

Problem 1.2.2: A toy rocket is fired straight up at an initial velocity of 30 m/s. The formula $h=30t-4.9t^2$ gives the height of the rocket above the ground t seconds after it is fired. How high does the rocket go? How long is it in the air?

Solution: Students should be encouraged to start by expressing the equation in the form $y = ax^2 + bx + c$ and then graphing it to find the vertex. They should note that the maximum they observe is a value of h and the time at which this occurs is the first number from the vertex, the value of t. They should then complete the square to place the equation in the form $y = (a - h)^2 + k$ and compare the vertex from this calculation, approximately (3.1, 45.9) with the one they observed when they graphed the function. In answering the question: "How long is it in the air?", the students are moving into the next concept. This, therefore, becomes a bridging activity. Once they accept the idea that the ground represents a value of 0 for h, they will be able to obtain this information from their graph. This inquiry is greatly aided by using the "trace" function of graphing calculators. It is the teacher's decision whether the students will get into a lot of the algebraic procedures for finding the zeros of the quadratic functions at this point.

This problem allows the students to apply some of the concepts they are studying. As mentioned, it also provides an interesting introduction into the next section. In addition, students have had the opportunity to work their way through the problem-solving steps. Using an experimental approach, they have investigated the properties of a quadratic function and its accompanying graph, and the necessary conclusions. They have seen the correlation between graphic and algebraic procedures in attempting to determine the information required. They have also studied the application of this material and had a glimpse of the future (i.e., the next section).

Before moving on to the next concept, don't forget to go back to the problem posed at the start of the unit. Usually the students won't let you. It provides a nice finish for this section.

TECHNOLOGY INTEGRATION

The capabilities of graphing calculators and computers have changed the way in which students can study this unit. Drawing graphs by hand on graph paper is rather laborious and when students perceive that they have to do a lot of it, they become less enthusiastic with the experimental approach. The technology is fun to use and provides for easy manipulation of the parameters.

The interactive BASIC computer program "Root Search" can be used to approximate the maximum height reached.

```
10      INPUT FIRST, LAST, CHANGE
20      PRINT "T", "H"
30      PRINT "____", "____"
40      FOR T = FIRST TO LAST STEP CHANGE
50      LET H=30*T-4,9*T^2
60      PRINT T, H
70      NEXT T
80      END
```

Input the first and last value of T to be used, and the increment. To refine the approximation of the height, modify the values for FIRST, LAST and CHANGE. The total travel time of the rocket is found by considering the equation $30t-4.9t^2$. This may be solved by factoring. The program may be used by approximating the time, T, at which H becomes 0. Focus on the domain in which H passes from positive to negative.

Adapted from "Computer Lessons in Algebra," The Ideas of Algebra, K-12, 1988 Yearbook of NCTM.

This application also lends itself readily to a spreadsheet. In fact, much of the experimentation studied in this section could also be studied with spreadsheets. The operator's manual or the Texas Instruments TI-81 calculator has an interesting application called "Solving an Equation."

See "Microcomputer Unit: Graphing Parabolas," Mathematics Teacher, December 1986 for BASIC program listing and a set of activities designed to help students examine the effects of A, B and C in the general equation, $y=ax^2+bx+c$, and to examine the relationship between the discriminant and the graph of the parabola.

Software:
Master Grapher (Addison-Wesley)
Computer Graphing Experiments (Addison-Wesley)
Green Globs and Graphing Equations (Sunburst)

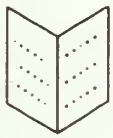
ELECTIVE SUGGESTIONS

In Mathematics 13 and 23 students have found the intercepts of the graphs of straight lines. Have the students use a calculator or computer to graph an equation given in the form $y=(x-h)^2+k$ and then find the intercepts both from the graph and from its equation. They don't necessarily have to use the quadratic equation techniques of factoring. They could isolate the x by subtracting the value of k, dividing by the value of a, taking the square root of both sides and then adding the value of h. This provides a bridge from previous material through this material and into the next concept.

Problem: In the comparable section of the Mathematics 20 Teacher Resource Manual, the following problem was given:

Determine the equation that has x-intercepts of 9 and 3 and a y-intercept of 18. The algebraic solution given in the Mathematics 20 Teacher Resource Manual may be too advanced for the average Mathematics 33 student, but there may be some other ways of attacking the problem with the use of a computer or graphing calculator.

RESOURCE CORRELATION



MMW 4: pp. 115–130
MM4: pp. 161–164; 169–172

CONCEPT 2

Students will be expected to demonstrate an understanding that finding the x-intercepts of the quadratic function $y = ax^2 + bx + c$ is equivalent to solving the quadratic equation $ax^2 + bx + c = 0$.

Students will be expected to:

2.1 use the quadratic formula to solve quadratic equations
  2.1.1 determine the x-intercepts of the quadratic function
  2.1.2 solve problems that involve quadratic equations
2.2 recognize and explain the relationship between the x-intercepts of the quadratic functions and the roots of their corresponding equations.

COMMENTS

In Mathematics 23 students solved quadratic equations by factoring. The preceding section discussed the quadratic function and began to hint at the relationship between the graph of the quadratic function and the solution of its corresponding equation. It's now time to address these relationships head on. Once again, the graphing calculator and computer can assist in the exploration of these similarities.

As the solution of quadratic equations by completing the square is not a stated objective, the development of the formula for the entire class should be considered an elective activity. However, the usefulness of the formula and the similarity of its results to those of factoring may be illustrated. As well, the usefulness of the formula in solving unfactorable equations should be highly touted. In drawing the relationships between the graph of the function and the roots of the equation, it will be helpful to show students the decimal equivalents of those solutions that have irrational solutions.

The characteristics of the graphs of those quadratic functions whose corresponding equations have no solutions should be illustrated graphically and algebraically.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 2.2: The teacher can have the students graph the functions from a carefully constructed exercise where the solutions to the corresponding equations can be obtained by factoring. The students can investigate the relationships between the graphs of the functions, the roots of the corresponding equations and the x-intercepts of the graphs. These can be summarized in chart form (see below) and then the students can have the opportunity to write some of their own conclusions. The activity can be expanded to include some functions whose roots can only be determined by using the quadratic formula. This could be a great cooperative learning, or group presentation, activity.

Here's an example of a possible chart and some other questions:

Functions	Equations	Equation Solutions	x-intercepts
1. $y=x^2-8x+12$			
2. $y=x^2-8x+16$			
3. $y=x^2-8x+14$			
4. $y=x^2-8x+20$ (etc.)			

a. How do the number of intercepts and the number of solutions relate?
b. Write a statement about the number of x-intercepts of a function and the number roots of the corresponding equation.
c. How many roots are there for question 4?
d. How many x-intercepts are there for question 4?
e. Write a statement about the relationship between quadratic functions that have no x-intercepts and the corresponding equations that have no solutions.

This activity provides an effective bridge between the previous concept (graphing) and this one (solving equations). Students writing their own conclusions about the characteristics of the various graphs and the relationships among the graphs, the x-intercepts and the roots of the corresponding equations, could be considered as a higher mental activity.

Problem 2.1 (return of the "rocket problem"): A toy rocket is fired into the air with an initial velocity of 30 m/s. The formula $h=30t-4.9t^2$ gives the height of the rocket above the ground t seconds after it is fired. How long is the rocket in the air?

Solution: Since the rocket leaves the ground at the start and returns to the ground, its height at these times is zero. Therefore the formula $h=30t-4.9t^2$ becomes the equation $-4.9t^2+30t=0$. The solutions to this are $t=0$ and $t=6.1$ (approx.). Therefore, the rocket was in the air for a total of approximately 6.1 seconds. Have the students graph the function and explore the region around the x-intercepts using the trace function of the calculator.

Once the exploration of the similarities of the graphs and equations have been completed, it's time to focus on the techniques for solving quadratic equations. Once the students appear to be confident in their abilities, try more complicated problems like the one following.

Problem 2.1.1: Consider two numbers whose sum is one. Which do you think is larger: the square of the smaller added to the larger or the square of the larger added to the smaller? Why?

Solution: They are the same as $x^2+(1-x)=(1-x)^2+x$.

Reprinted from delta k, April 1989, p. 52.

This problem could also be explored using a spreadsheet and by graphing both of the functions independently and then together on the same screen.

TECHNOLOGY INTEGRATION

The use of graphing calculators and graphing software has greatly changed the way in which this unit can be studied.

See MMW 4: p. 84.

See "Microcomputer Unit: Graphing Parabolas," Mathematics Teacher, December 1986 for BASIC program listing and a set of activities designed to help students examine the effects of a, b, and c in the general equation, $y=ax^2+bx+c$, and to examine the relationship between the discriminant and the graph of the parabola.

Software:
Factoring Algebraic Expressions (Mindscape) - review.

ELECTIVE SUGGESTIONS

A highly recommended elective topic is to study questions like the "toy rocket," which shows the interdependent nature of the function (vertex, maximum/minimum values, etc.) and the equation (x-intercepts, solutions).

Activity: Students can now use: $x-c$ is a factor of a trinomial function, $p(x)$ if and only if c is a root of $p(x)=0$. This fact also implies that when $\frac{a}{b}$ is a root of $f(x)=0$, then $bx-a$ is a factor. The following approach helps strengthen the connection between factoring and using the quadratic formula.

Consider the following example: Factor the expression $30x^2+49x-98$. Solve the equation $30x^2+98x-98=0$ to determine the factors of the given expression.

Then $x=\frac{-49\pm\sqrt{14161}}{60}=\frac{-49\pm119}{60}$

By simplifying for x and changing the equation to an expression equal to zero,

$$x = -\frac{14}{5} \quad \text{or} \quad x = \frac{7}{6},$$
$$5x = -14 \qquad 6x = 7,$$
$$5x + 14 = 0 \qquad 6x - 7 = 0.$$

then

$$30x^2 + 49x - 98 = (5x + 14)(6x - 7).$$

This reinforces the relationship between solving a quadratic equation and factoring.

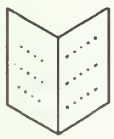
Working with the discriminant, $b^2 - 4ac$, is not included in the learner expectations. Therefore, its study should be considered an elective activity. There are some opportunities for students to do some additional exploration like those mentioned in Activity 2.2 involving comparisons among the graphs of various functions and their x-intercepts.

EXTENSION

The study of the quadratic inequalities could be studied as an option by some students. The focus should be primarily graphic with the aid of graphing calculators or the appropriate software.

RESOURCE CORRELATION

MMW 4: pp. 76–82; 85–86; 88–89
MM4: pp. 173–180



CONCEPT 2 (continued)

Students will be expected to demonstrate an understanding that finding the x-intercepts of the quadratic function $y = ax^2 + bx + c$ is equivalent to solving the quadratic equation $ax^2 + bx + c = 0$.

Students will be expected to:

2.3 solve equations with radicals and rational expressions whose solutions are the roots of quadratic functions

2.3.1 solve problems that involve equations with radicals and rational expressions whose solutions are the roots of quadratic equations.

COMMENTS

There is a continuing debate among mathematics teachers about where this concept should be taught. One school of thought says to teach it within the unit for which it is an application. This implies that those questions involving rational expressions should be taught in the rational expressions unit and those involving radicals should be covered in the radicals unit. Since basic quadratic equations are covered in Mathematics 23, this strategy is a possibility. However, it can be argued that questions that involve radicals and rational expressions are also extensions or applications of basic quadratic equations, and could be included in this unit.

The disadvantage of the first approach is that the coverage has to be limited to those questions that lead to quadratic equations which can be solved by factoring as students have not yet learned the quadratic formula. Its advantage is the immediacy of the concept to its root concept (i.e., equations with rational expressions to rational expressions). The disadvantage of the second approach is the lack of immediacy to its root concept, but it has the advantage of complete coverage by including any and all types of quadratic equations, factorable or not.

For the "fence-sitters" of the world, there is a third alternative. It is to discuss questions with radicals or rational expressions whose solutions do not involve quadratic equations in the units on radicals and rational expressions. In this unit questions whose solutions involve quadratic equations can then be studied in their entirety. This has the advantage of "recycling" or "spiralling" and thereby strengthening the approaches to solving radical equations and equations with rational expressions studied earlier. Its disadvantage is the time it may take to re-establish the techniques used earlier, and the overall strategy may appear fragmented to some students with parts of a concept taught in different sections at different times.

Regardless which approach is taken, the equations in this section should not become too difficult. The emphasis should be on demonstrating the application of the concept and not on solving overly difficult and complicated equations.

PROCESS/PROBLEM-SOLVING CONTEXT

Problem 2.3: Solve the equation $\frac{x^2+12}{x+3} = \frac{-7x}{x+3}$

Answer: The solution is -4. When both sides of the equation are multiplied by the factor $x+3$, the result leads to the quadratic equation $x^2+7x+12=0$ which, in turn, establishes -4 and -3 as values of x. However, -3 is a non-permissible value and is discarded as a root.

The significance of this type of question is that one of the solutions proves to be extraneous as it is in conflict with the non-permissible value ($x \neq 3$), which in turn reinforces the concept of non-permissible values. A problem like the next one shows a student that what might appear to be an easy question is not always what it appears to be.

An interesting approach to this question (as to any equation) is to explore the use of graphing calculators and computers. If students graph the left side as the function $y = \frac{x^2+12}{x+3}$ and on the same screen graph the right side as $y = \frac{-7x}{x+3}$, they will see them intersect at the point where $x = -4$. Because it may be difficult to see the intersection, have them rewrite the equation as $\frac{x^2-7x+12}{x+3} = 0$ and then graph it as the function $y = \frac{x^2+7x+12}{x+3}$. They will note that the graph is the straight line whose equation is $y = x+4$, which is exactly what they would obtain if they reduced the right side of the equation that they graphed. Also, they should note that the x-intercept is once again -4, which is the solution of the problem. Have the students try this approach on some of the other problems listed below or on ones they have already studied.

Problem 2.3: Solve the equation: $x - 4 = -\frac{1}{x}$

Answer: $x = 2 \pm \sqrt{3}$

Problem 2.3: Solve the equation: $x - 3 = \sqrt{5-x}$

Answer: $x = 4$

Once again the importance of this equation is to reinforce the idea that answers to radical equations must be checked to determine if any are extraneous.

As has been illustrated earlier in this manual, graphing calculators and computers can aid in these solutions by identifying possible solutions (the x-intercepts of the functions).

For more **Problems** see Problem Solving in Mathematics: Focus for the Future, p. 69, #25; p. 68, #16; p. 66, #14; p. 62, #8.

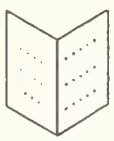
TECHNOLOGY INTEGRATION

See HM11: p. 62, "Computer Application: the Golden Ratio."

ELECTIVE SUGGESTIONS

Teachers should consult the Mathematics 20 Teacher Resource Manual, taking care to determine that the level of question is consistent with the expectations of Mathematics 33.

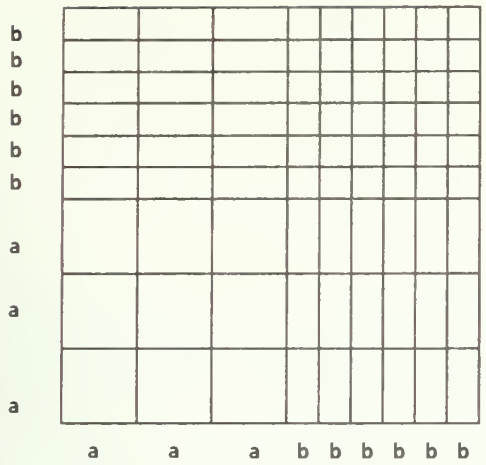
RESOURCE CORRELATION

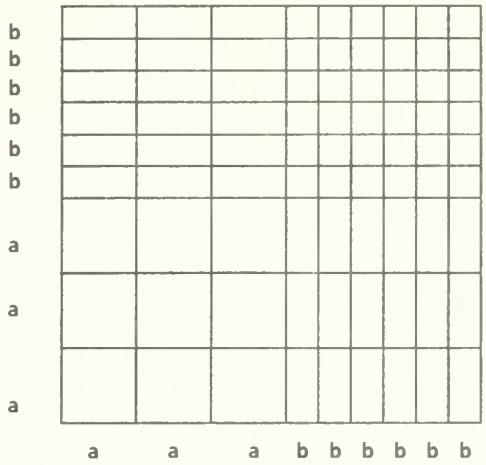


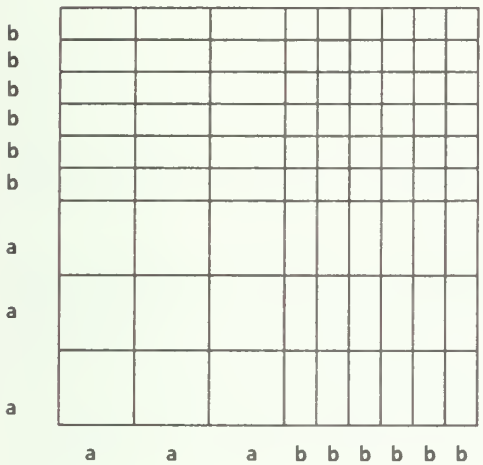
MMW 4:	pp. 92–93
MM4:	pp. 181–182

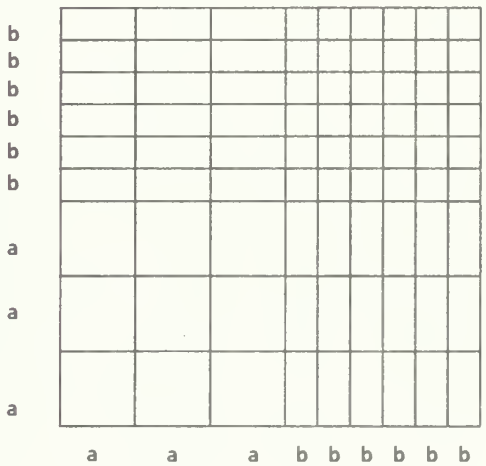
APPENDIX A

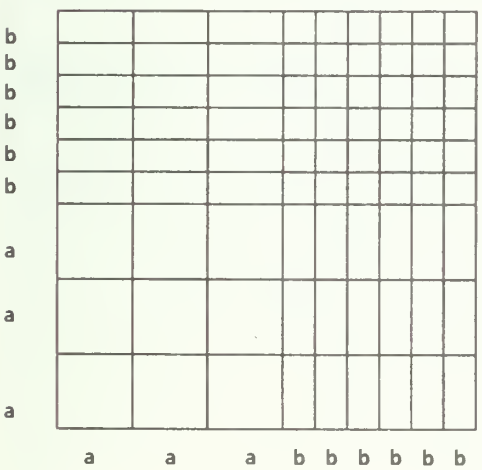
TEMPLATE: BINOMIAL GRID

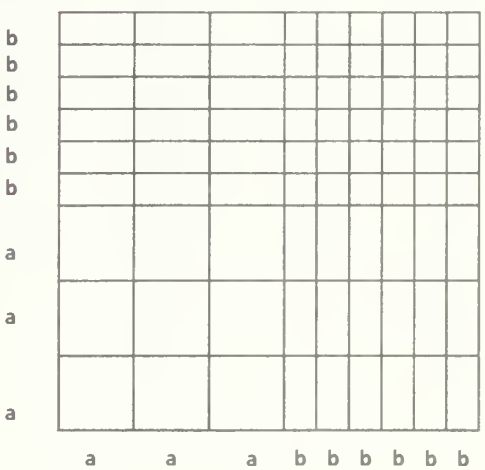




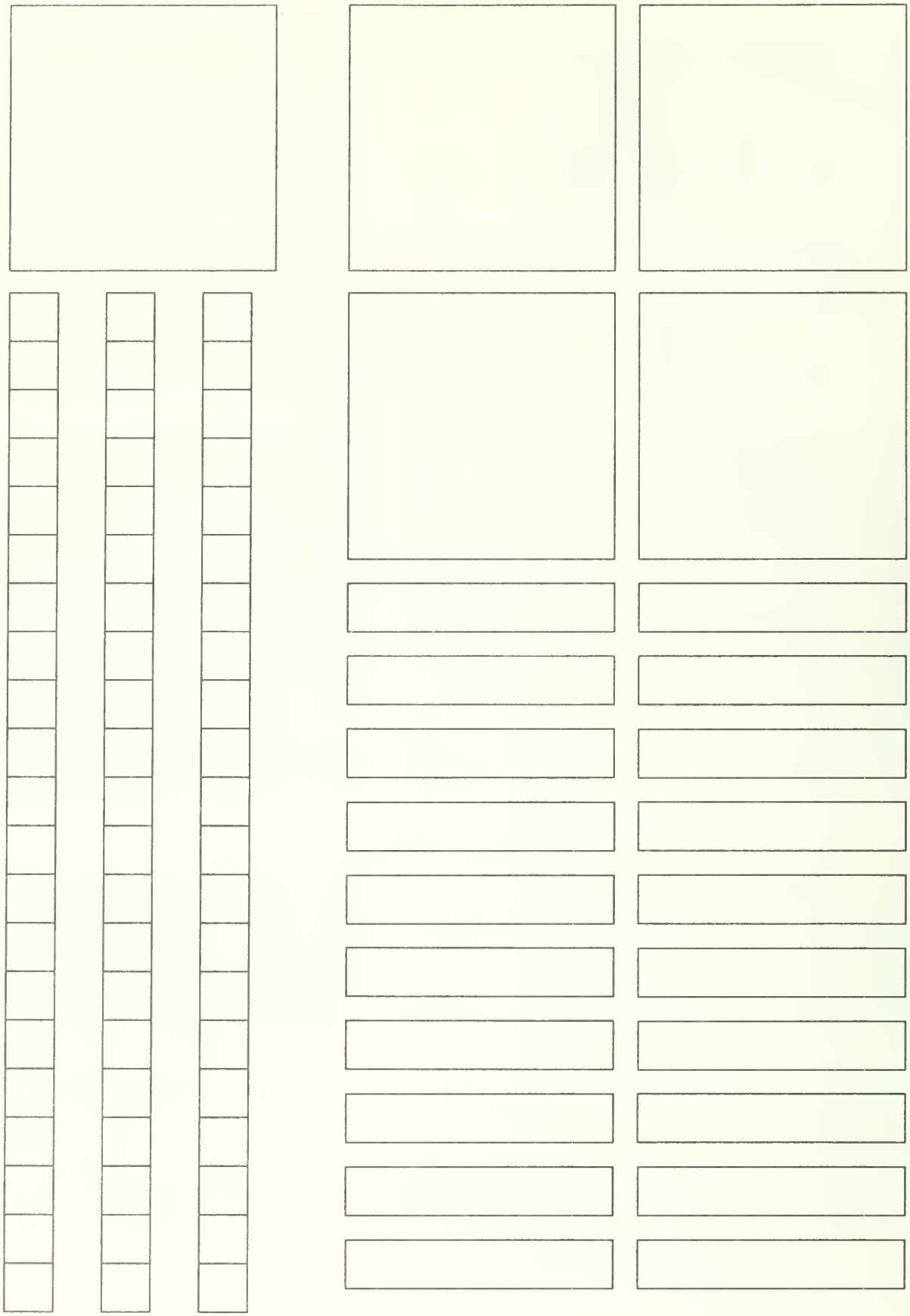








TEMPLATE: ALGEBRA TILES



APPENDIX B

SUPPORT RESOURCES

(A) Print

Activities for Implementing Curricular Themes from the "Agenda for Action". (Edited by C.R. Hirsch). Reston, VA: National Council of Teachers of Mathematics. ISBN: 0-87353-229-5. (Student) Provides activities in manipulatives, problem solving, "expanded" basic skills (e.g., exploratory data analysis, estimation), and calculator and computer use.

Activities for Junior High School and Middle School Mathematics: Readings from "Arithmetic Teacher" and "Mathematics Teacher". (Edited by K.E. Easterday, et al.). Simpson. Reston, VA: National Council of Teachers of Mathematics, 1981. ISBN: 0-87353-188-4 (Student) Provides activities in a wide range of mathematics topics.

Algebra Tiles for the Overhead Projector. (Howden, H.). New Rochelle, NY: Cuisenaire, 1985. ISBN: 0-914040-42-1 (Teacher). Describes using tiles on the overhead projector to demonstrate polynomials; the zero principle; adding, subtracting and multiplying polynomials; and factoring polynomials.

Algebra Tiles. Student Set.

Art and Techniques of Simulation - Teacher's Edition. (Gnanadesikan, M. et al.). Palo Alto, CA: Dale Seymour Publications, 1987. (Teacher). Provides teachers with ideas for implementing the suggestions identified within the student resource. Offers solutions to student exercises.

Curriculum and Evaluation Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics, 1989. ISBN: 0-87353-273-2 (Teacher). Provides a discussion on the direction of curriculum and evaluation in mathematics. The document discusses "standards" for curriculum in K-4, Grades 5-8 and Grades 9-12 and "standards" for evaluation of mathematical content, mathematical disposition and mathematical problem solving. There are several teaching and evaluation ideas throughout the resource.

Exploring Probability - Teacher's Edition. (Newman, C.M. et al.). Palo Alto, CA: Dale Seymour Publications, 1987. (Teacher). Provides teachers with ideas for implementing the suggestions identified in the student resource. Offers solutions to student exercises.

How to Evaluate Progress in Problem Solving. (Charles, R. et al.). Reston, VA: National Council of Teachers of Mathematics, 1987. ISBN: 0-87353-241-4 (Teacher). Discusses goals of problem-solving evaluation, evaluation techniques, organizing and managing an evaluation program, and using evaluation results.

Mathematics Dictionary. (4th ed.) (James and James). New York: Van Nostrand Reinhold, 1976. ISBN: 0-442-24091-0 (Teacher). Provides authoritative definitions of mathematics terms.

Problem Solving Challenge for Mathematics. Edmonton: Alberta Education, 1985. (Teacher.) Produced as a junior high school resource. Provides discussion of the nature of problem solving, strategies for teaching problem solving, a bank of problems, methods of evaluating problem-solving performance and a 36-item problem-solving bibliography.

Problem Solving in Mathematics: Focus for the Future. Edmonton: Alberta Education, 1987. (Teacher.) Produced as senior high school resource. Provides discussion of the nature of problem solving, strategies for teaching problem solving, project and computer-related problems, a bank of mathematics problems, methods of evaluating problem-solving performance and a 123-item problem-solving bibliography.

The Language of Graphs: A Collection of Teaching Materials. (draft). (Swan, M.). Nottingham England: Shell Centre for Mathematical Education, University of Nottingham, n.d. ISBN: 0-906126-11-8 (Student). Provides a wide range of graphing activities with emphasis on interpreting for meaning. Topics include modelling and interpreting with points, modelling situations, understanding distance–time graphs and interpreting complex graphs.

(B) Video

Of Dice and Men. (National Film Board). Available through ACCESS NETWORK Media Resource Centre or through the Regional and Urban Film Centres (see Appendix E). One 15-minute video. Provides a nice introduction to the development of probability and the uses of probability.

Trigonometric Functions I. (TV Ontario). Available through ACCESS NETWORK Media Resource Centre or through the Regional and Urban Film Centres (See Appendix E). A series of six 10-minute vignettes dealing with trigonometric functions. The six topics include: Similar Triangles, Solving Right Triangles, Angles on a Plane, The Sine Law, The Cosine Law and Applications of the Sine Law and Cosine Law. In Mathematics 20, this resource provides a good review of the trigonometric functions (the first two vignettes) and then provides a good introduction to the angles on a plane and the sine and cosine laws. In Mathematics 23, the first two vignettes will provide an alternative form of review when the unit is complete.

(C) Computer Software

Computer Graphing Experiments I. Addison–Wesley, (Apple II). Capable of graphing linear equations, parabola exponential–logarithm functions, inverse functions special functions such as absolute value and polynomials. "Linear Equations" option provides following sub-options: graphs of the form $y=mx$ and $y=mx+b$, and simultaneous equations. Students specify equation by inputting m and b. Extensive student worksheets provide activities in investigating effects of ma and b, parallel and perpendicular lines, writing equation in $y=mx+b$ form given $Ax+By+C=0$ form, two points or point–slope. Point of intersection of system of equations identified as ordered pair. Graphs of straight lines slightly ragged, appearance of perpendicularity problematic, no grid. Suitable for teacher demonstration, small group and individual student use. Teacher's guide with extensive set of student activity worksheets.

Master Grapher. (Waits, B. et al.). Reading, Mars: Addison–Wesley, 1988. (Apple II, IBM PC or PS/2, Macintosh) ISBN: 0-201-18246-7. Versatile utility graphing program capable of graphing a variety of functions, conics, parametric equations and polar form relations. Linear equations can be graphed in "Function Grapher" option. Suitable for exploring effects of m and b in $y=mx+b$, graphing systems of linear equations. Features include sliding vertical and horizontal lines, or cross-hairs that permit identification of x- and y-intercepts and point of intersection, change of viewing window, zoom in/out, grid display. Suitable for teacher demonstration, small group and individual student use.

Quantitative Literacy Software. American Statistical Association (Apple, IBM). Software to be used as a tool in examining statistics. The software supports the Quantitative Literacy Series of resources: Exploring Data, Exploring Probability, Art and Techniques of Simulation and Exploring Surveys. This software is particularly useful at the Mathematics 20 and 23 levels as it takes you through many simulations for probability.

SRA Computer Drill and Instruction, Mathematics, Level D. Willowdale, ON: Science Research Associates, 1981-1983. (Apple II, Recommended resource at the Junior High Level. Also available for IBM PC). A multi-disk package of drill exercises in computation, numbers and numeration, fractions, decimals, ratio and percent, measurement, pre-algebra and applications. Level appropriate for various copies in Mathematics 14, some in Mathematics 13. Does not provide answers or method of solving. May be used by students in unmanaged or teacher-managed form.

OTHER RESOURCES

(A) Computer Software

Algebra Drill and Practice I. (Detmer, R. et al.). Iowa City, IA: Conduit, 1981. (Apple II, IBM PC). Available from Conduit, University of Iowa, Oakdale Campus, Iowa City, IA 52242. $125 US (1988). Two-disk package provides drill and practice in a variety of topics including algebraic fractions, slopes and equations of line, percentages, simplifying linear algebraic expressions and word problems. "Slopes and equations of lines" option offers four types of problems: find slope given two points or given the equation in $Ax+By+C=0$ form and find equation of line given point and slope or two points. Must be expressed in $Ax+By+C=0$ form. "Percentage" option give percent problems of the form "20 is 30% of what number?" Provides example solutions, option of seeing solution to current problem. Typically, student inputs answer only, must solve mentally or on paper first. Teacher's guide included.

Algebra Drill and Practice II. (Detmer, R. et al.). Iowa City, IA: Conduit, 1982. (Apple II, IBM PC). Two-disk package provides drill and practice in simplifying integral exponent expressions, solving absolute value problems, linear equations and linear systems in two variables, factoring binomials and trinomials, finding roots of quadratic equations and solving linear inequalities. Step-by-step presentation of solution of examples and problems with which student has difficulty provided. Student inputs answer only, must solve mentally or on paper. Optional graphic solution of linear systems and inequalities provided by program following successful algebraic solution by student. Teacher's guide included.

Coordinate Math. St. Paul, MN: Minnesota Educational Computing Corporation, 1987. (Item No. A-192. ISBN: 0-87490-399-8. Apple II. Available from MECC, 3490 Lexington Avenue North, St. Paul, MN 55126). Consists of three educational games that require students to use their skills in problem solving and locating and using coordinate points on the Cartesian plane to accomplish a stated goal. Teacher's guide, with student worksheets, included.

Equation Math. St. Paul, MN: MECC, 1987. (Item No. 1-193. ISBN: 0-87490-405-6. Apple II). A versatile graphing program that offers three main options: "Exploring Equations," "Polygraph" and "Polar." The "Exploring Equations-Linear" option provides opportunity to explore effects of ma and b in $y=mx+b$. Equations of form $x=b$ and $y=b$ acceptable. Hard copy of screen contents optional. Automatic increment/decrement of parameters through cursor key action provided. "Quadratic equation" option also available.. "Polygraph" option capable of graphing numerous types of relations, including linear. Zoom in/out option provided. Display of ordered pairs that satisfy relation optional. Suitable for teacher demonstration and small group or individual student use. Teacher's guide, with extensive set of student worksheets, included.

Equations. (Ross, D.). Northbrook, IL: Mindscape, 1982. (Apple II, IBM PC, Commodore 64/128, Tandy 1000. Available form Mindscape, Inc., 3444 Dundee Road, Northbrook, IL 60062. $29.95 US (1988)). Provides very structured practice in solving equations of the form $AX+B=C$. Format is interactive step-by-step solution of the problem presented. Presentation of example solution and solutions of problems with which student has difficulty provided.

Equations II. (Appel, M. et al.). Northbrook, IL: Mindscape 1984. (Apple II, IBM PC, Commodore 64/128,Tandy 1000. Available from Mindscape, Inc., 3444 Dundee Road, Northbrook, IL 60062. $29.95 US (1988)). Provides very structured practice and remedial opportunities in solving equations of the form $Ax + B = Cx + D$. See Equation.

Equations III. (Ross, D.). Northbrook, IL: Mindscape, 1985. (Apple II). Provides very structured practice in solving equations of the form $Ax + B(Cx + D)$-$Ex + F$. See Equations.

Factoring Algebraic Expressions. Northbrook, IL: Mindscape. (Apple II, $29.95 US (1988)). Provides practice and remedial opportunities in factoring common factor problem, trinomials ("easier" :$a = 1$ and harder :$a \neq 1$) and difference of squares. Students must determine factors mentally or on paper, input as directed on screen. Program provides check by multiplication. Optional presentation of solution available.

Geometric preSupposer: Points and Lines. (Schwarta, J. et al.). Scarborough, ON: Sunburst, 1986. (Apple II. Available from Sunburst Communication, P.O. Box 3240, Station F, Scarborough, ON, M1W 9Z9. $132 (1989)). Provides extensive possibilities for drawing points, lines, segments, triangles, quadrilaterals, polygons, circles, bisectors of angles, and parallel and perpendicular lines. Measurement and/or calculation of length, perimeter, circumference, area or angles possible. Lines tend to be slightly ragged. Will take time for students to develop facility with program. Suitable for teacher demonstration, or geometry exploration by small group or individual students. Extensive teacher's guide, with student worksheets, included.

Geometric Supposer: Triangles. (Schwartz, J. et al.). Scarborough, ON: Sunburst, 1985. (Apple II, IBM PC, Tandy 1000). Provides extensive possibilities for exploring properties of triangles, which may be created by angle or side classification, or by construction according to SSS, SAS or ASA. Other figures such as line segments, circles, medians, altitudes, parallel and perpendicular lines, bisectors of angles and perpendicular bisectors of lines may be constructed. Measurement of lengths, perimeters, areas, angles and distance possible. Sum, difference, product, square and ratio of various measures also possible. See other comments, "Points and Lines." Extensive Teacher's guide, with student worksheets, included.

Graphing Linear Functions. (Berman, M. et al.). Northbrook, IL: Mindscape, 1984. (Apple II, IBM PC, Commodore 64/128, Tandy 1000. $29.95 US (1988)). Provides structured practice in graphing equations of the form $y = mx + b$ ("Easier" option) and "Harder" forms, for example, $Ax + By + C = 0$, $By = Ax + C$, ($B \neq 1$), etc. Students then determine three sets of ordered pairs that satisfy equation and plot by means of cursor key movements. Program then draws line in "Harder" forms option, students must first solve for y in step–by–step fashion.

Green Globs and Graphing Equations. (Dugdale, S. et al.). Scarborough, On: Sunburst. 1986. (Apple II, IBM PC, Tandy 1000. $87 (1989)). A versatile graphing program. "Equation Plotter" option is a utility program in which a wide variety of forms of equations can be input and graphed. "Linear" sub-option provides opportunity to explore effects of m and b in $y = mx + b$; input in $Ax + By + C = 0$ form; investigate parallel and perpendicular lines and $y = b$, $x = b$ forms; etc. Systems of equations may also be graphed, although coordinates of intersection point not given. Other modes of "Equation Plotter" permit such things as rational, polynomial, absolute value quadrated and exponential functions and identifying zeros. Suitable for teacher demonstration and small group or individual student use. "Linear and Quadratic Graphs" option provides practice in writing equations. Program draws graph, student must write equation that satisfies graph. Extensive teacher's guide, with student worksheets, included.

Percents. Baltimore: Media Materials, 1983. (Apple II, except II c). Available from Media Materials, 2936 Remington Avenue, Baltimore, MD 21211. $49.95 US (1988) Provides practice and remedial opportunities in solving percent problems. Teacher's guide, with student worksheets, included.

Ratios and Proportions. Baltimore: Media Materials, 1983. (Apple II, except II c). Provides practice in solving ratio and proportion problems. Teacher's guide, with student worksheets, included.

Solving Quadratic Equations. (Ross, D. et al.) Northbrook, IL: Mindscape, 1982. (Apple II, IBM PC, Commodore 64/128, Tandy 1000. $29.95 US (1988)). Provides structured practice in solving quadratic equations. Two options: "Easier (A = 1)" and "Harder (A = 2 or 3)." Students presented with quadratic equation, which must be factored mentally or with pencil-and-paper. Program checks by multiplying. Students must then solve for each of the factors in step-by-step manner. Program demonstrates solution for questions student has difficulty with.

Spreadsheets for Mathematics and Science: Appleworks Sampler. St. Paul, MN: MECC, 1987. (Item No. T-260. ISBN: 0-87490-415-3, Apple II). Provides practice in learning and using spreadsheets (specifically, Appleworks) through its application to mathematics and science problems. Requires time. Teacher must know how to use spreadsheet before introducing to students. Extensive teacher's guide, with student activity sheets, included.

(B) Print

A Sourcebook of Applications of School Mathematics. Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics. Reston, VA: National Council of Teachers of Mathematics, 1980. ISBN: 0-87353-164-7. (Teacher). Contains a number of articles that discuss use of applications in mathematics, and provide specific examples of application in advanced arithmetic, algebra, geometry, trigonometry and probability.

Activities from the "Mathematics Teacher". (Edited by E.M. Maletsky et al.). Reston, VA: National Council of Teachers of Mathematics, 1981. ISBN: 0-87353-173-6. (Student). Provides activities in a number of mathematics topics.

Algebra Before & After: 1. (Gregory, J.). Palo Alto, CA: Creative Publications, 1987. ISBN: 0-88488-641-7 (Student). A workbook of student developmental and practice activities related to topics in polynomials (exponents, writing expressions, multiplying and factoring), writing and solving linear equations and systems of equations, coordinate geometry and graphing, square roots and the Pythagorean Theorem. Emphasis is on pattern development and generalization.

Algebra Before & After: 2. (Gregory, J.). Palo Alto, CA: Creative Publications, 1987. ISBN: 0-88488-642-5 (Student). A workbook of student developmental and practice activities related to topics in radicals and rational exponents, polynomials (multiplying, factoring, number pattern generalizations) rational expressions, relations and functions (graphing, zeros) and quadratic equations. Emphasis is on pattern development and generalization.

Applications in School Mathematics: 1979 Yearbook. (Edited by S. Sharron et al.). Reston, VA: National Council of Teachers of Mathematics, 1979. ISBN: 0-87353-139-6 (Teacher). Numerous examples provided of applications in mathematics in this collection of articles. Extensive bibliography also provided.

Canada's Food Guide Handbook. Ottawa: Department of National Health and Welfare, 1988. ISBN: 0-662-11947-9 (Teacher). Provides resource material suitable for use in ratio and proportion problems and projects.

Classroom Ideas from Research on Secondary School Mathematics. (Dessart, D. and Suydam, M.). Reston, VA: National Council of Teachers of Mathematics, 1983. ISBN: 0-87353-207-4 (Teacher). Presents a variety of ideas for teaching algebra and geometry in the secondary school. Discusses research on teaching, cognitive development, attitude and affective factors, problem solving, the use of calculators and computers, and testing and prediction.

Computers in Mathematics Education, 1984 Yearbook. (Edited by V.P. Hansen et al.). Reston, VA: National Council of Teachers of Mathematics, 1984. ISBN: 0-87353-210-4 (Teacher). Discusses computing from standpoints of its challenge, impact and perspective as a diagnostic tool. Discusses programming as a means of teaching mathematics.

Computing and Mathematics: The Impact on Secondary School Curricula. (Edited by J.T. Fey). Reston, VA: National Council of Teachers of Mathematics, 1984. ISBN: 0-87353-212-0. (Teacher). Report of a major conference. Topics include mathematics and the new information technology, impact of computing on algebra, geometry, calculus and discrete mathematics and algorithmic methods and prospects and strategies for change in school mathematics.

Creative Geometry: Investigations on the Microcomputer. (Teacher's Guide). (Bell, A. et al., South Notts Project). Nottingham, U.K.: Shell Centre for Mathematical Education, University of Nottingham, University Park, Nottingham, NG7 2RD, n.d. Discusses activities in creating geometric patterns and designs by creating and using programs like those in LOGO.

Descartes' Dream: The World According to Mathematics. (Davis, P. et al.). Boston: Houghton Mifflin, 1986. ISBN: 0-395-43154-9 (Teacher). Discusses the nature of mathematics, the impact of computing on mathematics, the impact of both on society. Eclectic in style, highly readable.

Estimation and Mental Computation: 1986 Yearbook. (Edited by H.L. Schoen et al.). Reston, VA: National Council of Teachers of Mathematics, 1986. ISBN: 0-87353-226-0 (Teacher). A collection of articles that provide rationale and strategies for developing and evaluating estimation and mental computation skills.

Everybody Counts: A Report to the Nation on the Future of Mathematics Education. (Board on Mathematical Sciences [and] Council (U.S.)). Washington, D.C.: National Academy of Sciences, 1989. ISBN: 0-309-03977-0 (Teacher). Discusses mathematics, mathematics education and their relation to society. Relevant to Canadian educators.

Imaginative Ideas for the Teacher of Mathematics, Grades K-12: Ranucci's Reservoir. (Ranucci, E.R., Edited by M.A. Farrell.). Reston, VA: National Council of Teachers of Mathematics, 1988. ISBN: 0-87353-257-0 (Teacher). A variety of topics covered in this collection of articles. Often suitable for extension activities.

Learning and Teaching Geometry, K-12: 1987 Yearbook. (Edited by M.M. Lindquist et al.). Reston, VA: National Council of Teachers of mathematics, 1987. ISBN: 0-87353-235-X (Teacher). Contains a number of articles that deal with topics such as issues in teaching geometry, problem-solving activities and relating geometry to other mathematics topics.

Math Projects for Young Scientists. (Thomas, D.A.). New York: Franklin Watts, 1988. ISBN: 0-531-10523-7 (Student). Discusses how to choose, research and present a mathematics project, and briefly describes 96 suggested student projects. Designed for the capable, interested student of mathematics. Varying degrees of background mathematics sophistication required.

Mathematics: A Human Endeavour. (2nd ed.) (Jacobs, H.R.) New York: W.H. Freeman, 1982. ISBN: 0-7167-1326-8 (Student). An innovative textbook exploring a variety of mathematics topics.

Mind Tools: The Five Levels of Mathematical Reality. (Rucker, R.). Boston: Houghton Mifflin, 1987. ISBN: 0-395-46810-8 (Teacher). Explores the nature of mathematics as number, space, logic, infinity and information. Very non-routine but readable discussion of the nature of mathematics.

Mindstorms: Children, Computer, and Powerful Ideas. (Papert, S.). New York: Basic Books, 1980. ISBN: 0-465-04629-0 (Teacher). Describes the development of LOGO and its potential use by children to develop mathematical thinking and understanding.

Problem Solving in School Mathematics: 1980 Yearbook. (Edited by S. Krulik et al.). Reston, VA: National Council of Teachers of Mathematics, 1980. ISBN: 0-87353-162-0 (Teacher). Contains a number of articles describing various aspects of problem solving.

Recent Revolutions in Mathematics. (Stwertka, A.). New York: Franklin Watts, 1987. ISBN: 0-531-10418-4 (Teacher). Provides a readable overview of recent activity in the field of mathematics. Topics include the geometry of the universe, the science of prediction, number theory, fractals, chaos and catastrophe theory, and computers.

Student Math Notes. (Edited by E.M. Maletsky). Reston, VA: National Council of Teachers of Mathematics, 1987. ISBN: 0-87353-244-9. Contains a number of articles with interesting non-routine student activities. See for example: "Pi: The Digit Hunt" (Coffield, P.), pp. 39-44. Presents a number of historical notes and activities regarding the approximation of pi.

Teaching Mathematics: A Sourcebook of Aids, Activities and Strategies. (2nd ed.) (Sobel, M. et al.). Englewood Cliffs, NJ: Prentice Hall, 1988. ISBN: 0-13-894148-3 (Teacher). A sourcebook for secondary mathematics teachers. Covers a wide variety of topics.

Teaching Statistics and Probability: 1981 Yearbook. (Edited by A.P. Shulte et al.). Reston, VA: National Council of Mathematics, 1981. ISBN: 0-87353-170-1 (Teacher). Provides discussion on rationale, samples of existing programs, classroom activities, teaching and learning specific topics, applications, statistical inference, Monte Carlo techniques, using computers, and an extensive bibliography and list of projects.

The Art of Problem Posing. (Brown, S.I. et al.). Philadelphia: Franklin Institute, 1983. ISBN: 0-89168-052-7 (Teacher). Provides strategies for and numerous examples of problem posing, designed to encourage open-ended problem solving.

The Ideas of Algebra, K-12: 1988 Yearbook. (Edited by A.F. Coxford et al.). Reston, VA: National Council of Teachers of Mathematics, 1988. ISBN: 0-87353-250-3 (Teacher). Covers a number of topics related to school algebra, including ideas and issues, readiness, equations and expressions, problem solving, computer and calculator use, and teaching ideas.

The Mathematical Experience. (Davis, P. et al.). Boston: Birkhauser, 1981. Penguin Books ed. (1983). ISBN: 0-14-022456-4 (Teacher). Explores widely the nature of mathematics: history, schools of philosophical thought, symbolic nature, views of some contemporary mathematicians and teachers, teaching, and learning and uses. Insightful, written in highly readable style.

The Nature of Mathematics. (4th ed.). (Smith, K.J.). Monterey, CA: Brooks/Cole, 1984. ISBN: 0-534-02806-3 (Student). Innovative, wide-ranging textbook. Explores the nature of a number of mathematics topics, such as inductive and deductive reasoning through patterns and sets, calculators and computers, numbers, algebra, geometry, and statistics and probability.

Thinking Mathematically. (rev.ed.). (Mason J. et al.). Wokingham, Eng.: Addison-Wesley, 1985. ISBN: 0-201-10238-2 (Teacher). A structured, activity-based book of discussion and open-ended questions designed to help develop ability to think mathematically. Attends to the emotional aspects of working with mathematical problems. Very readable and engaging.

(C) Journals

Computers in Education. Moorhead Publications, Ltd., 1300 Don Mills Road, North York, Toronto, ON, M3B 3M8. (Teacher). Provides information on microcomputers and software. Canadian focus. Articles, items discuss computing and programming uses in education, including mathematics education. Published 10 times a year.

delta-K. Mathematics Council of the Alberta Teachers' Association. (Teacher). (Address correspondence to J.B. Percevault, editor, 2510 - 22 Avenue South, Lethbridge, Alberta, T1K 1J5. Articles include discussions of issues in mathematics education, general approaches and styles in mathematics teaching and learning, and specific teaching ideas.

delta-K. Special Edition: "Mathematics for Gifted Students." Vol. 27, No. 3, April 1989. Presents articles on special topics in mathematics, plus lists of more than 350 problems with solutions and more than 140 books of mathematical puzzles, problems, instruction and applications.

For the Learning of Mathematics. FLM Publishing Association, 4336 Marcil Avenue, Montreal, Quebec H4A 2Z8. (Teacher). Articles deal with a variety of mathematics-related topics, including issues in mathematics education, philosophy of mathematics and research on teaching and learning mathematics. Intended for a wide range of mathematics educators. Published three times a year.

Mathematics Teacher. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, Va 22091. (Teacher). A journal intended for the secondary mathematics teacher. Scope includes mathematics education issues, teaching ideas, innovative secondary level mathematics and resource reviews. Published 10 times annually.

School Science and Mathematics. School Science and Mathematics Association, Inc., 126 Life Science Building, Bowling Green University, Bowling Green, OH 43403. (Teacher). Articles discuss issues in mathematics and science education, research on teaching and learning mathematics, science and teaching ideas.

The Computing Teacher. International Council for Computers in Education, University of Oregon, 17787 Agate Street, Eugene, OR, 97403-9905. (Teacher). Emphasizes teaching and learning about computers, teaching with computers, teacher education and the impact of computers on curricula. Published nine times a year.

APPENDIX C

SAMPLES OF STUDENT EVALUATION IDEAS

Evaluating Problem Solving

Rating	Exceptional (5)	Proficient (4)	Satisfactory (3)	Limited (2)	Poor (1)
Category					
Understands problem.	Complete understanding of problem.	Most of problem understood.	Part of problem understood.	Limited understanding of problem.	Complete misunderstanding of problem.
Develops plan to solve problem.	Plan would lead to a logical conclusion.	Mostly appropriate plan; may have to be revised.	Partially appropriate plan; based on an incomplete understanding of problem.	Limited attempt to develop a plan.	No attempt to develop a plan; no logical conclusion could be reached.
Arrives at a solution.	Logical solution to problem; appropriate answer.	Solution mostly appropriate; minor error.	Some error in answer, but solution direct result of error in plan.	Major error in solution.	No answer given; illogical solution.
Evaluates solution.	Solution evaluated accurately based on information provided.	Evaluation mostly complete and appropriate.	Satisfactory attempt to evaluate solution and use data collected.	Limited attempt to evaluate solution or use data.	No attempt to evaluate solution or use data collected.

Note: Student excused –
Unexcused absence – rating 0
Assignment not done or handed in – rating 0

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Checklist

Student: **Date:**

_____ 1. Likes to solve problems
_____ 2. Works cooperatively with others in the group
_____ 3. Contributes ideas to group problem solving
_____ 4. Perseveres – sticks with a problem
_____ 5. Tries to understand what the problem is about
_____ 6. Can deal with data in solving problems
_____ 7. Thinks about which strategies might help
_____ 8. Is flexible – tries different strategies if needed
_____ 9. Checks results for accuracy
____ 10. Can describe or analyse results – come to an appropriate conclusion or decision

Rating Scale

Student: **Date:** _____

	Frequently	Sometimes	Never
1. Selects appropriate solution strategies	_____	_____	_____
2. Accurately implements solution strategies	_____	_____	_____
3. Tries a different solution strategy when stuck (without help from the teacher)	_____	_____	_____
4. Approaches problems in a systematic manner (clarifies the question, identifies needed data, plans, solves and checks)	_____	_____	_____
5. Shows a willingness to try problems	_____	_____	_____
6. Demonstrates self-confidence	_____	_____	_____
7. Perseveres in problem-solving attempts	_____	_____	_____

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	Student 1	Student 2	Student 3	Student 4	Student 5
Date: Name Action observed					
Confidence: Initiates questions.					
Is sure answers will be found.					
Helps others with problems.					
Other/note:					
Flexibility: Solves problems in more than one way.					
Changes opinion when given a convincing argument.					
Other/note:					

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SELF-ASSESSMENT DATA FROM STUDENTS

Rating Scale

Use the following questions to help you look back and describe your thinking as you worked toward a solution to the problem (or worked through the investigation, assignment, project).

1. What did you do when you first saw the problem? What were your thoughts?
2. Did you use any problem-solving strategies? Which ones? How did they work?
3. Did you try one approach that didn't work and have to stop and try another approach? How did you feel about this?
4. Did you complete the assignment (investigation, project)? How do you feel about this?
5. Did you check your work in any way to ensure the original assignment was carried out correctly? Did you feel sure it was correct (satisfactory)?

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FOCUSED MARKING SCALE

7. EXCEPTIONAL

- content: ideas, organization and development command attention and respect
- an impressive style evident from word choice, usage and sentence structure
- mechanical skill accurate and effective in relation to purpose
- accurate spelling

6. SUPERIOR

- content: most ideas are significant; well organized and developed
- sentence structure and word choice clear and effective; accurate usage
- mechanical skills relatively error free in relation to purpose
- relatively free from spelling errors

5. COMPETENT

- content: several significant ideas; minor problems in organization and development
- minor problems in word choice, and/or sentence structure
- frequent mechanical errors
- few spelling errors

4. MARGINAL

- content: a few relevant ideas; some evidence of organization but with deficiencies in development
- many flaws in word choice and/or sentence structure
- many distracting mechanical errors
- numerous, distracting spelling errors

3. UNSATISFACTORY

- content: insufficient number of ideas; disorganized, inadequate development
- many flaws in word choice and/or sentence structure
- many distracting mechanical errors
- numerous, distracting spelling errors

2. MAJOR DEFECTS

- content: lacking in ideas, or organization and/or development
- serious deficiencies in word choice and/or usage and/or sentence structure
- extensive mechanical errors
- extensive spelling errors

1. INSUFFICIENT DATA AND/OR OFF TOPIC

Developed by Program Evaluation and Language Arts Team, Calgary Board of Education.

Suggestions for Further Reading:

Board of Education for the City of Etobicoke (1987). Making the Grade: Evaluating Student Progress. Prentice-Hall Canada Inc. Scarborough, Ontario.

Joint Mathematical Board Shell Centre for Mathematical Education (undated). Problems with Patterns and Numbers: An O-level Module.

National Council of Teachers of Mathematics (1988). How to Evaluate Progress in Problem Solving. Reston, Virginia.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, Virginia.

APPENDIX D

USING WRITING TO LEARN MATHEMATICS

C.L. Nahrgang and B.T. Petersen

Writing has a place in a mathematics class. It provides a new and interesting mode of communication between the student and teacher, as Watson (1980) reported:

This two-way conversation has become beneficial to the class. The students realized I hear them and care. They seem to have looked inside themselves and to have seen what they could do to help solve their mathematical problems. Many of their grades improved.

Written assignments in mathematics classes also afford students the opportunity to organize their thoughts and, at the same time, improve their writing skills (Johnson 1983).

The purpose of this article is to show that writing can be used in mathematics to enhance learning. The act of writing gives students the opportunity to formulate, organize, internalize and evaluate concepts. Emig, who has done extensive investigation of the thinking processes involved in writing, argues that writing "represents a unique mode of learning – not merely valuable, not merely special, but unique" (Emig 1977). The idea that writing is a powerful aid to learning has also found much support in cognitive psychology. For example, Bruner (1966) states, "the more we know about the manner in which it [writing] can aid thought...[the more it] leads me to put language at the centre of the stage in considering the nature of intellectual development."

Our experience indicates that the most effective method of using writing to help students learn mathematics is through the use of journals. These in-class writing exercises offer students the opportunity to work informally and personally on mathematical concepts, using their own language and real-world experiences. They can plan with a variety of concepts without the fear of the formal evaluation of their writing and their mathematical skills. Our experience in college-level classes, from algebra through calculus, makes us feel confident that these short but powerful writing exercises enhance learning. The journal goes beyond rote learning and challenges the students to use intellectual skills. Myers relates writing to individual learning: "When doing this simple activity [writing], students will be forced to relate information from the lecture to what they already know and to organize and synthesize it so that the concept becomes their own" (Myers 1984).

The Journal

The journal is a diary-like series of writing assignments. Each assignment or entry is a short written response to an instructor's question, statement or set of instructions. All responses are written in prose rather than in the traditional mathematical style of numbers and equations. The students are directed to use an expressive writing style so that their responses are spontaneous and all their thoughts are recorded. Figure 1 is an example of a journal entry made by a student in technical writing in response to the following assignment: "Describe a real-life, non-mathematical analogy to the rule $a^{-n} = 1/a^n$."

Journals have two basic functions. First, they allow students to proceed at their own rate and to converge on an understanding of mathematical concepts using their own experiences. Second, they provide teachers a unique diagnostic tool; the writings of students immediately reveal areas of confusion and expose misunderstandings of mathematical concepts.

Journals were introduced into our mathematics classes at Michigan Tech with a minimum of difficulty. These classes were conducted over a 10-week period. Each class included about 10 unannounced quizzes that accounted for 20 percent of a student's final grade. By substituting 20 journal-writing sessions for these quizzes, we gave our students two opportunities a week for writing in their journals. However, the flexibility – one of its advantages – of the journal-writing technique permits the instructor to devote extra assignments to more difficult mathematical concepts.

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$a^{-n} = \frac{1}{a^n}$ is an inverse rule so, if any power is raised to the negative it becomes

a fraction, also making it positive.

You inverse what it says, you switch it around.

So what in life is switched around or inversed (which means the same thing)?

What can I turn around in my life – yet keep it the same; to mean the same (yet it is non-mathematics)?

- *relationships*
- *feelings*
- *communication – the way I say something or communicate*

The English language and communication could easily apply to this rule. We may change or turn phrases around yet keep their meaning the same.

Figure 1

Most journal entries require three to seven minutes, depending on the type of assignment. Our experiences have shown that as students become more familiar and comfortable with this technique, their responses become better organized and thus easier to read. In all classes, we found that students' experiences in writing in journals led to definite improvements in their ability to organize their responses.

The method of evaluating journals become an important question. Initially, grading each journal seemed to be most appropriate, with the average of all journal grades being a part of the final grade for the course. We immediately realize that this method restricted the potential of this learning tool. By assigning grades, we were indirectly telling the students how they should process concepts, which was defeating one of the important attributes of journal writing – individualized learning and discovery. However, students seemed to deserve some type of credit for their efforts. After considering several alternatives, we decided to give one or two points toward the next major examination on the basis of the students' comments about concepts. The most appropriate responses to mathematical concepts were given the maximum extra credit allowable. A lack of effort received no credit. In addition, we often added appropriate comments (e.g., "excellent," "good," "incomplete" or "please revise"). We found that students appreciated the teacher's comments as much as they did the extra credit. Finally, all mathematical errors were corrected. Although these errors were a secondary consideration, they were brought to the students' attention.

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An Analysis

Journal entries demonstrate individual learning activities. Various types of assignments were made that coincided generally with the levels specified in Bloom's taxonomy (Bloom, Hastings and Madaus 1971.) During a section on factoring, students were asked to do the following journal assignment: "Discuss the following statement: 'Factoring and finding a product are reverse processes.'" The objective of this particular assignment was to have students collect ideas, internalize the concept of factoring and examine its significance in finding a product. This task was assigned early in the study of factoring and finding products of polynomials. Observe how each of the following students begins to understand the process of factoring in a personal frame of reference. Student A wrote:

Factoring and finding a product are reverse processes. They result as reverse processes but the way you go about it isn't reverse because the opposite of multiplication is division. The equations are interchangeable though and can be called.

At first, this response appears to be nonsense. Notice, however, the student's attempt to link multiplication ad division to finding a product and factoring. This link is important because it gives this student a bridge to familiar concepts, multiplication and division.

Even in the response of student B, a marginal student, we can see the cognitive value of writing:

Factoring and finding a product are reverse processes. Factoring I guess is to make finding the answer much easier. I really am not sure how much it helps because I can't see much point in it. I guess that it puts the problem in simpler terms.

First, the student restates the assignment and then works toward a basic but correct evaluation. The importance of this student's response can be seen in the last sentence. The student discovers one of the main purposes of factoring, to simplify an expression so that it can be used in further operations.

Student C shows an understanding of the mechanics of the two processes but demonstrates no insight into the concept or purpose of factoring.

I believe that factoring and finding a product are reverse procedures. When you find a product of two binomials that is multiply:

$(x + 2)(2x - 1) = 2x^2 + 3x - 2$
When you factor a trinomial you end up with two binomials:

$2x^2 + 3x - 2 = (x + 2)(x - 1)$

This response also illustrates a difficulty with evaluating statements. The more equations and mathematics we see in a response, the less we know about students' understanding of a concept. For this reason we always encourage our students to answer with words and sentences rather than with equations.

In the following example, notice how student D collects ideas on finding a product and factoring, then, in the third sentence, attempts to answer self-generated questions, with a satisfactory conclusion.

Combining factors to find a product is an application of algebra that allows factors to be combined into either monomials, polynomials or trinomials, or whatever. Factoring is the process of taking monomials, polynomials, binomials, trinomials or whatever and reducing them to the terms that multiply together to form them. Thus, factoring is taking polynomials apart, while finding the product is putting them together. Actually the two are very similar in the processes used. One uses primarily multiplication, the other division.

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The student collects, internalizes and finally evaluates. For the student, the journal is a progression toward the understanding of a concept. For the teacher, the journal is a record of a student's thinking and thus an excellent diagnostic tool.

Given the same assignment, some students focused their thoughts on isolating a common factor. For this method of factoring, students sometimes confused division with the isolation of a common factor. They did not grasp the idea that division is actually contained within the process of isolating a common factor. The success or failure of students in recognizing this difference is clearly demonstrated by the responses of students E and F, respectively:

Factoring is sort of dividing $(3x^2 + 6x)$ when you factor out the $3x$ you get $3x(x + 2)$. To multiply $3x(x + 2)$ you do the opposite and you get $3x^2 + 6x$ again. So finding a product and factoring are the same simply reversed.

When factoring you remove all common factors from an equation. These are removed usually by dividing them out of the whole equation. Now to get back to the original equation from the factored one, you have to multiply the factored equation by the numbers and variable, etc., that were removed. This shows that factoring and finding the product....

Student E doesn't separate the two ideas as well as student F. In the second sentence student F clearly understands that division is a part of the process of isolating the common factor.

A subsequent assignment in the same class required students to translate the concept of finding a product and factoring to a non-mathematical event. Students responded to the following prompt, "Think of a non-mathematical relationship that is analogous to the process of finding a product and factoring." Student G gave one of the better responses:

One relationship similar to finding a product vs. factoring is that of taking a carburetor apart and then putting it back together. When the carburetor is together it is difficult to clean and repair. Therefore it is disassembled to make it easier to work with. This is analogous to factoring. When the parts of the carburetor are in working condition, they must be put back together before they will work as desired. This is analogous to finding a product.

This journal assignment differs from the other assignments because students need to compare non-mathematical experiences to what is being learned in class. The response of student G demonstrates an overall understanding of factoring and finding a product by comparing these processes to the disassembly and assembly of a carburetor. He understands their relationship to each other and is able to connect the two processes. His analogy also illustrates that factored and product forms have different mathematical uses. In this assignment the student had to collect information from both mathematical and non-mathematical experiences and then connect them logically. This example shows that in one short journal a student can use a variety of intellectual skills, such as synthesis, interpretation, translation, analysis and evaluation. We believe that with continued use of these skills, students grow both in the understanding of mathematical concepts and in the ability to express that understanding.

Conclusion

We adapted the technique of journal writing to instruction in mathematics classes. We attempted to promote an improved internalization of mathematical concepts and to see how journal writing could be incorporated in these classes. This study examined students' test scores, attitudes and writing (the latter by the Writing Apprehension Test) as well as student and teacher evaluations. The data have not established any strong relationships among the various attributes examined. An examination of journals, however, clearly indicates that students used their journals to think about solving problems

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associated with mathematical concepts. In addition, most of the students' responses to a questionnaire indicated that students viewed journals as a worthwhile inclusion in mathematics classes.

Any conclusion derived from this study must be viewed and extended cautiously. They are specific to a particular isolated setting. The study has led to guarded conclusions and has generated many more questions than it has answered. The investigating team hopes that these efforts are more of a "first step" rather than a "stopping place" and that further study will establish more definitive relationships between writing and mathematics.

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Mathematics Teacher, December 1986,
National Council of Teachers of Mathematics.

EXPLORATIVE WRITING AND LEARNING MATHEMATICS

S.Z. Keith

Short explorative writing assignments can transform the mathematics classroom into a dynamic and exciting learning laboratory. In explorative writing, students explore their knowledge about a topic by writing what they know about it in their own language. Then, to refine their ideas further, they share reactions with other students and the teacher. In spite of the significant benefits of using such writing assignments, the teacher may feel uncomfortable with the more interactive environment, with the challenge of designing writing assignments and with figuring out what to do with the results. This article illustrates how explorative writing assignments can help to expose and identify learning problems and offers some assignments that address these problems. This article briefly mentions some techniques that I have found helpful for working with the results and comments on the benefits of working with these assignments.

How Writing Assignments Expose Learning Problems

I use writing assignments continually in my classes with students of all levels. In one class of trigonometry students, a typical assignment had the immediate purpose of addressing the difficulties many students have in transferring from the visual to the verbal mode. The question was, "What is a reference angle?" An illustration and adequate definition of the concept are shown in Figure 1.

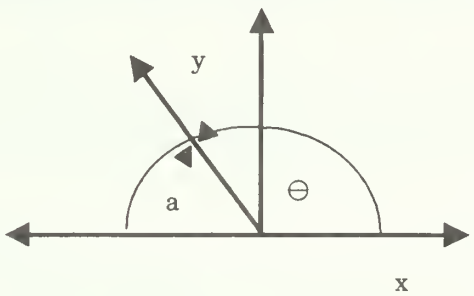

Fig. 1. An illustration and adequate definition of a reference angle.

DEFINITION: A reference angle for a given angle θ is an acute angle x that is formed by the X-axis and the terminal side of θ (x is a reference angle for the angle θ).

Whereas all the students could compute reference angles, only a few could adequately define the term. The students' difficulties fell into the following patterns:

1. Some students merely gave an example (which of course was a start):

   - An angle like if you have 400 degrees on the unit circle your reference angle would be 40 degrees.

2. Others struggled for precision or failed to recognize the conditions of necessity and sufficiency that the definition requires:

   - Another angle that is the same as your first angle but in simpler terms.

   (The words *same, first* and *simpler* are vague, and what the "simpler terms" are is the essence of this definition).

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- The acute angle with the same terminal side.

- An angle closer to the x–axis than the original angle.

3. Some tried to define simply by stating how the construction is used:

   - The angle that has to measure up supplementary with the angle given that tells which quadrant it is in.

   - The angle that makes it easier to compute the trig functions for the first angle.

4. Frequently students were either vague or said as much as possible, with the objective of maximizing partial credit on homework or a test:

   - An angle for which measurements of less than 90 are made in relationship to the x–axis only. The angle is smaller than the original and has equivalent trig functions, like 30, 60, etc., etc.

5. Surprisingly, very few students drew pictures, and when they did, the drawings were neither labelled nor referenced. Although drawing pictures may not be considered a part of writing definitions, students find it much easier to discuss or write about a concept if they have learned to draw an accurate picture first. Most students first grasp a concept visually or tactilely rather than verbally.

Another type of difficulty students have lies in using conventions of notation. Asking students to define a polynomial over the real numbers produces surprising results, even in higher-level calculus classes:

- A polynomial is $ax^2 + bx + c$.

  (This is only a second-degree polynomial. This student may never before have realized the need for the notation, $a_1, a_2, \ldots$.).

- f(x) is a polynomial over the reals if $f(x) = a^1{}_x + a^2{}_x + \ldots$.

  (This student has remembered only that some notation is "up" and some "down").

Often the students presume that the reader already knows the answer and so an explanation or analysis is unnecessary. The assignment to describe the procedure for computing the determinant of a $x \times 3$ matrix generated the following responses:

- Take the row–1, column–1 position and multiply it by the determinant of the 2 x 2 matrix....

- Do not use the whole expansion, use co-factors, which is just telling what sign we give it. A plus or a minus times the minor.

The situation becomes more critical still when the question is not merely to state a definition or an algorithm but to write a proof. Then the problem becomes the students' lack of understanding of what constitutes a proof. The examples that follow are from an upper-level abstract algebra course:

- Question: Prove that if H and K are normal subgroups of a group G, H, K is a normal subgroup of G.

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Answer: Since H and K are subgroups, H K is normal by a theorem.

- Question: Prove that lcm (m, n) = mn/gcd(m, n).

Answer: We must have that lcm(m, n) - mn/gcd(m, n) because assume that this is not true. But that can't be because we know that lcm(m, n) = mn/gcd(m, n). Therefore this is a true statement.

Much energy in college teaching is dedicated to teaching advanced students to do proofs – questioning how to do them and wondering whether a proof is actually possible in certain situations. The algebra students whose work is shown in the foregoing may not belong in an upper-level course; they have reached this point only by mastering some well-digested problems. Their lack of understanding underscores the need for writing assignments at the earliest levels. Furthermore, the failure of many of our students to successfully create proofs or to write expositions raises questions about what they successfully construe or read and about the development of their learning skills. Frequently, students completely avoid reading the text and get by with methodically working exercises.

Activities That Address the Problems

Most of my explorative assignments are in-class summaries or assignments to be completed overnight; they are given to help the students understand a mathematical concept. They include the following types of exercises:

1. *Summaries.* Summaries can reveal where a class stands or serve as an introduction to a concept.

   - Right now, what is your notion of a "function?" Be as precise as possible: we say "y = f(x) is a function of x if...."

   - What's wrong with this logic?

     (Example of a reducing-fractions problem.)

   - What is a "radian," and why do we prefer this notion to that of "degree?"

2. *Visual image translation.* These experiences integrate visual and verbal understanding.

   - Describe the graph of this function as if you were explaining it to a friend over the telephone.

   - How are the graphs of $y = 1/x^2$ and $y = x^2$ related? How could you predict the behaviour of the second from that of the first?

   - The class is divided into pairs. One student describes a graph in language that is as accurate and precise as possible while the other draws the graph. No peeking allowed on either side.

3. *Synopsizing tactics for solving a problem.* These exercises enhance students' awareness of problem-solving techniques.

   - Describe how to combine fractions to obtain one term. Do not merely give an example.

   - What is the difference between combinations and permutations and by what clues do you distinguish problems involving the one or the other?

4. *Giving an algorithm.* Students practise describing steps in algorithmic thinking.

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- Write an explicit procedure that helps you to solve distance–time–rate problems.

5. *Giving a definition or stating a theorem.* Such verbalizations enhance the precision of students' terms and theorems.

6. *Communicating thoughts to a specific audience.* These exercises give students the experience of translating one explanation into another.

   - Many of you did not understand the concept on page 12 of the text. Rewrite that page as you would have liked to have seen it written.

   - In a few pages, write a "crib sheet" for your best friend, who has fallen seriously behind in this course. List important concepts, definitions and theorems and briefly explain any methods of this unit on which the class might be tested.

7. Inventing a problem. Students learning to create their own illustrations through such exercises.

   - You live in a family with three sisters, four brothers, two dogs and 17 cats. Create several problems that deal with combinations and permutations involving those numbers. Give answers and explain what allowed you to recognize that the problem was to be solved by the techniques you used.

8. *Generalizing a concept.* These exercises help to strengthen students' grasp of the overview of mathematics.

   - Mathematicians write definitions when a clear need arises. In your own study of polynomials, what definitions might you write to simplify talking about polynomials? Where could your definitions be useful?

   - Where in "real life" have you come across "absolute value?"

9. *Group projects.* Such projects enhance confidence and productivity by having students work collaboratively. These activities allow students to exchange ideas and to pursue the development of an idea in greater depth than they can when working alone. Since students enjoy this activity, it is a good way to introduce writing assignments.

   - Pass out a few transparencies and markers and have groups compete in writing an accurate statement of a theorem or an explanation of a procedure. Show the results on the overhead projector.

   - Have the groups anticipate a theorem. For instance, "A squirrel runs into a box and out again. It may enter through two doors and exit through three, so it can pass through the box in as many as six different ways. A very important principle is at work here – one that doesn't involve squirrels. Can you state it in its greatest generality?" Or, "Now that you have seen my explanations for finding maxima and minima visually, can you explain how the first derivative can indicate where a relative maximum or minimum occurs? If you can, you will have stated the *first derivative test.*"

Techniques and Benefits of Using Writing Assignments

Giving writing assignments, of course, does not in itself automatically generate effective teaching and learning; the teacher must respond to the writing. It is useful to approach writing as part of a process leading to completion in the future rather than as a final product; this approach shifts the perspective from what is wrong to how it could be made more accurate. Attempting to achieve this shift has led me to use several techniques.

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1. *Discussion.* I put the students' efforts on transparencies, and show them on the overhead projector, encouraging discussion or giving my own reactions. In conformity with the NCTM's *Curriculum and Evaluation Standards for School Mathematics*, which state that mathematics should be considered as a form of communication, every effort is made to understand each student's attempt. Any inclinations on the part of students to ridicule or be embarrassed generally subside when it becomes clear that I am trying to understand their thought processes and how their definitions might be improved. Something interesting and worthwhile is usually evident in every student's attempt, and, interestingly, students' answers often go awry in similar ways. The reactions of other students are useful in explaining what was intended; most of us have found that students sometimes have a mysterious ability to interpret the teacher's intent to other students.

2. *Peer evaluation.* Papers can be exchanged and evaluated according to specific criteria (see Figure 2 for a sample worksheet). Such peer evaluation gives students an opportunity to see papers from a perspective closer to that of the teacher.

3. *Revision.* After the need for precision has been shown through peer evaluation, students may be asked to rewrite their definitions. Frequently, the entire exercise is repeated on another day. Students may think they understand a concept when the teacher shows the corrections on their work, but teachers can create no better learning opportunity for students than having them reconstruct their own answers in the correct way.

4. *Anticipation.* Quite often, the writing assignments generate lively discussion and can be used to introduce a topic; for instance, the assignment to define a degree can be followed by a discussion of the usefulness of radians.

These exercises are short and well worth the time they take. They teach strategies for maximizing understanding; they develop trust and improve participation and collaboration in and out of the classroom. Most students appreciate the chance to be heard and understood. From their point of view, the elegance of the text or the lectures may merely be obfuscating. Students are generally receptive to the discussion of class writing projects and to doing extra work at home because they can see the payoff in their enhanced understanding of the material. Students report that they feel they read and retain material more easily as mathematical conventions become familiar to them.

In the test setting, accuracy and formal quality are of course expected, but the students develop this quality by themselves as a result of their in-class experiences. Test results are frequently improved because detailed notice of a student's work sharpens the student's attention to detail.

I have found that I do not need to grade grammar and mechanics. My objective is to get closer to how students think, and continually correcting spelling and grammar tends to drive students into using a smaller vocabulary and taking fewer risks. In the students' writing generally, their control of grammar and mechanics is closely linked with their ability to use logic. The issue is simply whether they are adequately communicating with me, and the explicit grading, when done at all, is based on this criterion and goes very quickly.

Above all, the teacher can emphasize the fact that the writing of definitions, theorems and later, proofs is a delightful experience. The aspect of elegance in our mathematical language can and should be shared by our students at every level.

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DEFINITION AND THEOREM WORKSHEET

DEFINITION WORKSHEET: NAME:_____

DEFINE:

5 – The student has given a complete definition, with clear and accurate vocabulary and notation, appropriate mathematical vocabulary and minimum redundancy.

4 – Minor problems with clarity, notation, vocabulary or wordiness are evident.

3 – The basic concept is clear (student seems to know what is being defined), but student exhibits significant confusion in notation or vocabulary.

2 – The basic concept shows significant confusion, with major problems in notation or vocabulary.

1 – The definition does not relate to the concept in question.

THEOREM WORKSHEET: NAME:_____

STATE:

5 – The student has given a complete theorem, with clear and accurate vocabulary and notation, a complete hypothesis and conclusion, appropriate mathematical vocabulary and minimum redundancy.

4 – Minor problems arise with clarity, notation or vocabulary; or minor incompleteness is found in the hypothesis or conclusion.

3 – The basic concept of the theorem is clear (student seems to understand the theorem), but significant confusion or error is seen in vocabulary or notation.

2 – The basic concept of the theorem shows significant confusion about the hypothesis or conclusion, with major errors in notation or vocabulary.

1 – The theorem does not relate to the concept in question.

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Hillocks, George, Jr. "The Responses of College Freshmen to Three Modes of Instruction." *American Journal of Education* 89 (1981): 373–95.

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Mathematics Teacher, December 1988,
National Council of Teachers of Mathematics.

APPENDIX E

SHARING TEACHING IDEAS

THE TWELVE DAYS OF CHRISTMAS AND PASCAL'S TRIANGLE

This lesson was used with various algebra and geometry classes just before Christmas vacation. For the lesson, I had plenty of chalk on hand in bright colours, especially "tree green" and "ornament red, yellow and blue." My Christmas "lecture" is just brimming with mathematics, and it keeps the students on pins and pine needles for an entire class period.

The sources of my inspiration were the words of the traditional Christmas carol, "The Twelve Days of Christmas," and Janet Hartman's activity, "Figurative Numbers" (*Mathematics Teacher,* January 1976, pp. 47–50).

The night before my Christmas lecture, I assign a few students to bring in the words to "The Twelve Days of Christmas." At the beginning of the lesson, I write a list of gifts and the days they are given, making it clear that each day the true love brings new gifts and duplicates all the preceding gifts:

1st day—1 partridge
2nd day— 2 turtledoves (and the above)
3rd day—3 French hens (and all of the above)
4th day—4 colly birds
5th day—5 golden rings
6th day—6 geese a-laying
7th day—7 swans a-swimming
8th day—8 maids a-milking
9th day—9 lords a-leaping
10th day—10 ladies dancing
11th day—11 pipers piping
12th day—12 drummers drumming

"Did you ever wonder how many gifts 'the beloved' received on a particular day—how many gifts she had accumulated by a particular day?" I ask my students. We examine some specific values. For example, on the fourth day she received 10 gifts, bringing the four-day cumulative total to 20 gifts.

Their interest is piqued, and I begin to write a few rows of Pascal's triangle on the board (in green, of course—it's a tree). Then I add some white squiggly "tinsel" arrows for the unenlightened, to show how two neighbouring entries are added to get entries for the next row (see Figure 1).

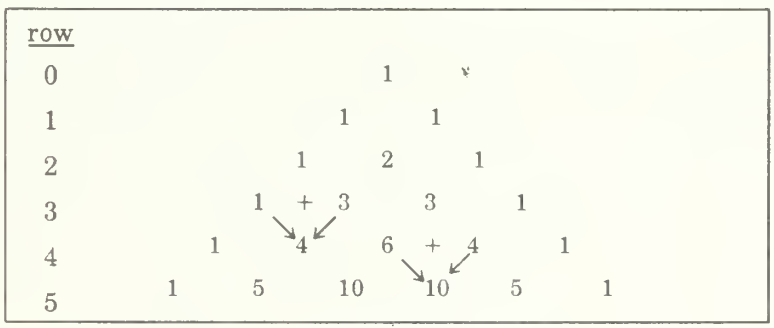

Fig. 1. Pascal's triangle

From *Mathematics Teacher*, Vol. 7519 Dec. 1982.

Then we observe one or more of the nice patterns in Pascal's triangle. For example, each row starts and ends with 1. The counting numbers are very obvious in the diagonal next to the ones. The row entries are the powers of 11 until we get to the fifth row. The rows add up to the powers of 2. The coefficients of $(a+b)^n$ are also row entries. We can find the odd integers by adding consecutive pairs of integers in the diagonal of counting numbers. The squares of integers are found in the next diagonal by adding consecutive pairs of entries, as shown in Figure 2.

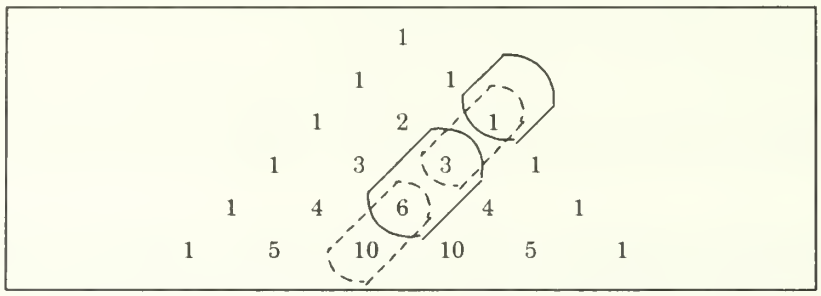

Fig. 2. Pascal's triangle showing squares 1, 4, 9, 16

Returning to seasonal matters, we draw the arrays of the triangle numbers (Figure 3). They make nice little trees too. The triangle numbers are also found in Pascal's triangle, along a diagonal that can be seen as a string of ornaments. (For clarity, only one diagonal is marked in Figure 5.)

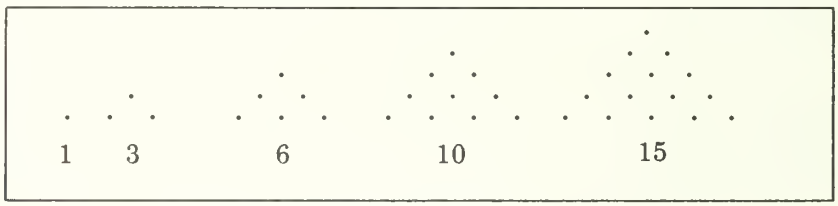

Fig. 3. Triangle numbers 1, 3, 6, 10, 16

The pyramid numbers are a three-dimensional version of the triangle numbers. The first 1, and the second is 4, obtained by using a base layer of three objects and putting 1 on the next layer. The third pyramid number, 10, is obtained by using the next larger triangle number, 6, for a base, a second level of 3, and a third level of 1 (Figure 4).

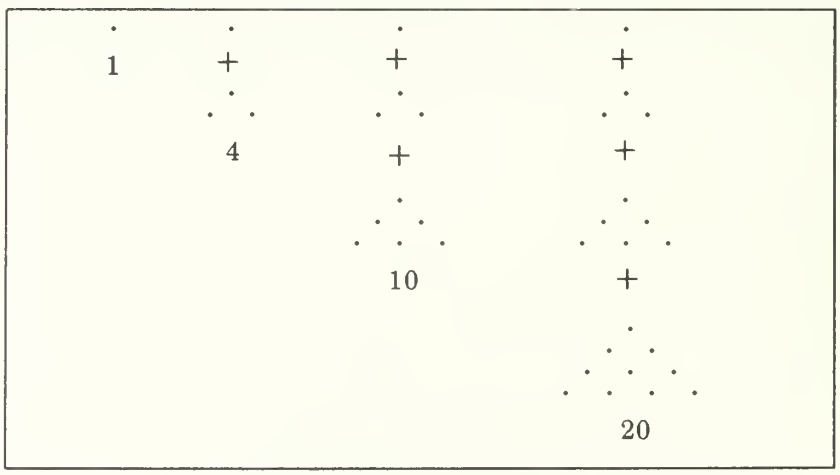

Fig. 4. Stack the triangular numbers to get the pyramid numbers 1, 4, 10, 20

Continuing in this way, we can find a pattern for the pyramid numbers that appear in Pascal's triangle. We now have another string of ornaments for the tree! (See Figure 5.)

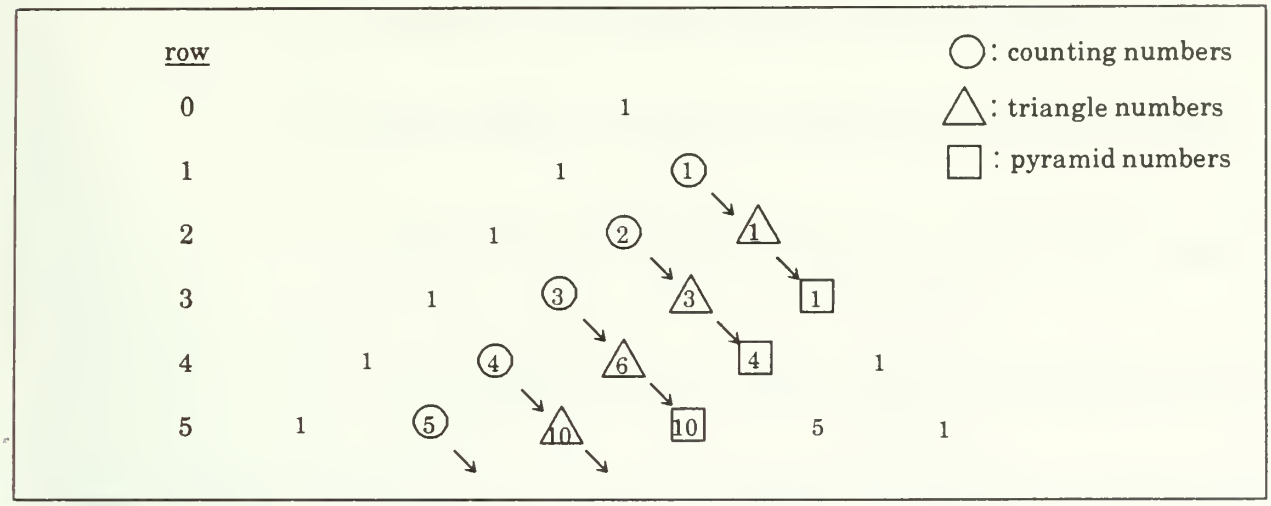

Fig. 5

So what does all this have to do with 'The Twelve Days of Christmas"? On the first day of Christmas, the beloved received 1 gift, on the second day $2+1=3$ gifts, on the third day, $3+2+1=6$ gifts and so on. Notice that 1, 3, 6, . . . are the triangle numbers in Pascal's triangle. On the first day of Christmas, she had accumulated only one gift; on the second day she received three more gifts for a total accumulation of four gifts. On the third day she received six more for a total accumulation of 10 gifts. The daily accumulations of gifts are 1, 4, 10, ..., which are the pyramid numbers found in Pascal's triangle. To find how many gifts she received on the twelfth day, we can extend the triangle to the thirteenth row, go over three places and read 78 (Figure 6). What is the total number of gifts accumulated by the end of the twelfth day? Follow the arrows in Figure 6 to the fourteenth row and find 364.

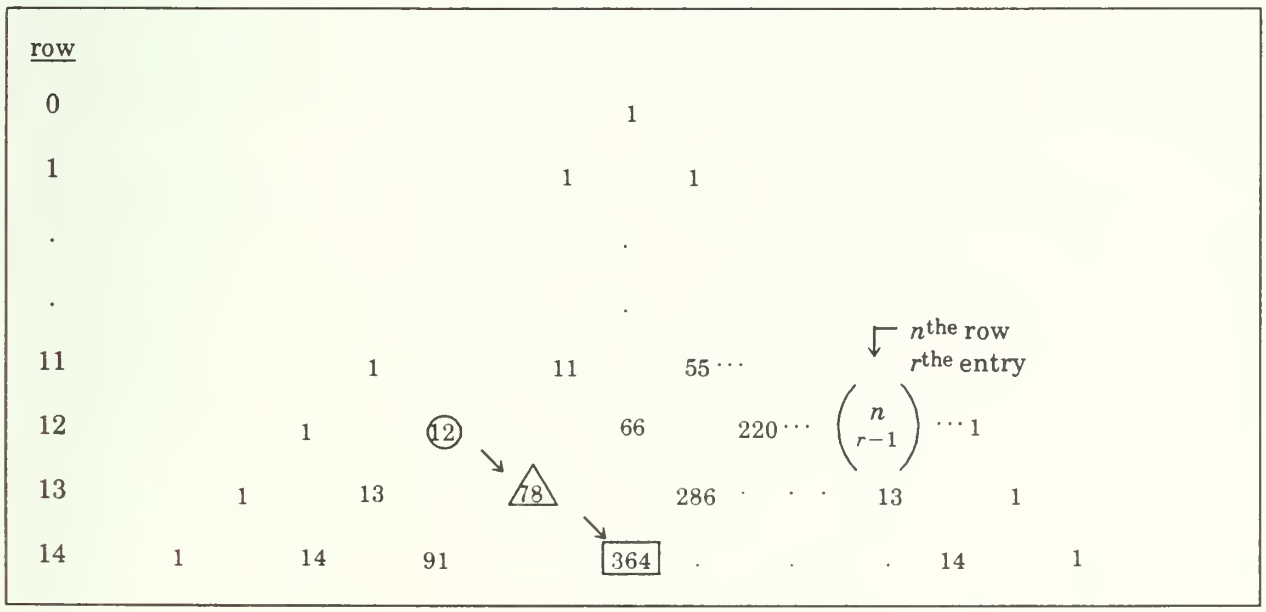

Fig. 5

The real mind-boggler occurs if a class has been briefed in combinations, $\binom{n}{r}$.

Note that in order to get the rth entry in the nth row, we should compute $\binom{n}{r-1}$.

So the number of gifts, g, received on the dth day is given by the formula

$$g=\binom{d+1}{2},$$

and the formula for total gift accumulation, t, on the dth day is

$$t=\binom{d+2}{3}.$$

The richness of mathematical relationships seemed to impress and surprise my students. I wish I had more days before Christmas to let them discover more mathematics like this. They seemed to enjoy it as much as I did.

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Cincinnati, OH 45215

APPENDIX F

GUIDELINES FOR EFFECTIVE MEDIA INTEGRATION

There is a growing collection of media resources that support the expectations of the Senior High Mathematics program. As with any resource, films and videos should be selected with due consideration for the expectations of the program and the needs of your class. Specific films and videos are noted throughout the teacher resource manual.

The following questions should be considered when selecting media resources for use in the classroom.

- Does it support the expectations of the program?
- Does it meet individual student needs and learning styles?
- Does it reinforce, enrich or extend the program?
- How appropriate is the timing?
- Does the medium demonstrate abstract ideas?
- How well does the medium fit into existing curricula and teaching/learning needs?
- How clear are the educational aims and objectives?
- What kind of preparation is required?
- How are teaching and learning evaluated?
- What forms of follow–up are required?

Resource/Media Centres

There are 12 resource/media centres in Alberta that carry films and videos that support the Senior High Mathematics program. Each centre publishes its own catalogue, listing the resources in its collection. These centres operate as libraries, lending out audiovisual materials for specified periods. For more information contact the local resource/media centre:

(A) Regional Resource Centres

Zone 1
Zone One Regional Film Centre
P.O. Box 6536
10020 – 101 Street
Peace River, Alberta
T8S 1S3
Telephone: (403) 624–3187

Zone 2/3
Central Alberta Media Services
(CAMS)
2017 Brentwood Boulevard
Sherwood Park, Alberta
T8A 0X2
Telephone: (403) 464–5540 or 467–8896

Zone 4
Alberta Central Regional Educational Services
(ACRES)
County of Lacombe
Box 3220
5140 – 49 Street
Lacombe, Alberta
T0C 1S0
Telephone: (403) 782–5730

Zone 5
South Central Alberta Film Federation
(SCAFF)
Westmount School
Box 90
Wheatland Trail
Strathmore, Alberta
T0J 3H0
Telephone: (403) 934–5028

Zone 6
Southern Alberta Regional Film Centre
(SAFRC)
McNally School
P.O. Box 845
Lethbridge, Alberta
T1J 3Z8
Telephone: (403) 320–7807

(B) Urban Resource Centres

County of Strathcona
Learning Resource Service
2001 Sherwood Drive
Sherwood Park, Alberta
T8A 3W7
Telephone: (403) 464–8235

Calgary Separate School Board
Instructional Materials
6220 Lakeview Drive S.W.
Calgary, Alberta
T3E 6T1
Telephone: (403) 246–6663

Edmonton Public School Board
Learning Resources Centre
Centre for Education
One Kingsway
Edmonton, Alberta
T5H 3G9
Telephone: (403) 429–8320

Edmonton Catholic Schools
Curricular Resources
St. Anthony's Teacher Centre
10425 – 84 Avenue
Edmonton, Alberta
T6E 2H3
Telephone: (403) 439–7356

Red Deer Public School Board
4747 – 53 Street
Red Deer, Alberta
T4N 2E6
Telephone: (403) 343–1405

Calgary Board of Education
Education Media
3610 – 9th Street S.E.
Calgary, Alberta
T2G 3C5
Telephone: (403) 294–8540

Medicine Hat School District
Instructional Materials Centre
601 First Avenue S.W.
Medicine Hat, Alberta
T1A 4Y7
Telephone: (403) 526–1323

APPENDIX G

ALBERTA CONSUMER AND CORPORATE AFFAIRS

Calgary	301 Centre 70; 7015 Macleod Trail South Box 5880, Postal Station A Calgary, Alberta, T2H 2M9 297–5700
Edmonton	3rd Floor, Capilano Centre 9945 – 50 Street Edmonton, Alberta T6A 0L4 427-5782
Fort McMurray	Fort McMurray Provincial Building 5th Floor, West Tower 9915 Franklin Avenue Fort McMurray, Alberta T9H 2K4 743–7231
Grande Prairie	Grande Prairie Provincial Building 10320 – 99 Street, Box 7 Grande Prairie, Alberta, T8V 6J4
Lethbridge	300 Professional Building Bag Service 3014 740 – 4 Avenue South Lethbridge, Alberta, T1J 4C7 381–5360
Medicine Hat	303 Provincial Building 770 6 Street S.W. Medicine Hat, Alberta, T1J 4J6 529–3535
Peace River	Provincial Building 9621 – 96 Avenue Bag 900, Box 9 Peace River, Alberta, T0H 2X0 624–6214
Red Deer	2nd Floor, Provincial Building 4920 – 51 Street Red Deer, Alberta, T4N 6K8 340–5241
Alberta Consumer and Corporate Affairs	Box 1616 Edmonton, Alberta, T5J 2N9

APPENDIX H

ALBERTA EDUCATION MATHEMATICS CONSULTANTS

Grande Prairie Regional Office

Hugh Sanders
5th Floor, Nordic Court
10014 – 99 Street
Grande Prairie, Alberta T8V 3N4
Phone: 538–5130
Fax: 538–5135

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Dick Daly
Devonian Building – West Tower
11160 Jasper Avenue
Edmonton, Alberta T5K 0L2
Phone: 427–2952
Fax: 422–9682

Art Peddicord
Devonian Building – West Tower
11160 Jasper Avenue
Edmonton, Alberta T5K 0L2
Phone: 427–2952
Fax: 422–9682

Red Deer Regional Office

Ron Babiuk
3rd Floor West, Provincial Building
4920 – 51 Street
Red Deer, Alberta T4N 6K8
Phone: 340–5262
Fax: 340–5305

Calgary Regional Office

Pat McLaughlin
Room 1200, Rocky Mountain Plaza
615 Macleod Trail, S.E.
Calgary, Alberta T2G 4T8
Phone: 297–6353
Fax: 297–3842

Lethbridge Regional Office

Gary Hill
Provincial Building
200 – 5th Avenue, South
Lethbridge, Alberta T1J 4C7
Phone: 381–5243
Fax: 381–5734

MATH 33
REVERSE TEXTUAL CORRELATION
NELSON PUBLISHING COMPANY
MATH MATTERS BOOK 4 (MM4)
(Pilot Edition)

Page	Unit	Learner Expectation Number	TRM Page
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65–67	Polynomials and Rational Expressions	1.1, 1.2	170–173
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169–172	Quadratic Functions and Equations	1.1, 1.2	196–201
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MATH 33
REVERSE TEXTUAL CORRELATION
GAGE
MATHEMATICS FOR A MODERN WORLD BOOK 4 (MMW4)

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57–59	Polynomials and Rational Expressions	1.3, 1.4, 1.5	174–176
59–60	Polynomials and Rational Expressions	1.1, 1.2	170–173
62–64, 69–72	Polynomials and Rational Expressions	1.3, 1.4, 1.5	174–176
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92–96	Radicals and Exponents	1.1, 1.2, 1.3	130–133
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115–130	Quadratic Functions and Equations	1.1, 1.2	196–201

MATH 33
REVERSE TEXTUAL CORRELATION
NELSON
MATH MATTERS BOOK 4 (MM4)

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186–192	Trigonometry	1.3, 1.4, 1.5	156–158
193–195	Trigonometry	1.6	159–160
208–212	Trigonometry	2.1	161–162
216–219	Trigonometry	2.2	163–165
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MATH 30
REVERSE TEXTUAL CORRELATION
ADDISON-WESLEY
MATHEMATICS 12

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279–281	Exponential and Logarithmic Functions	3.1, 3.2, 3.3, 4.1, 4.2	85–87, 88–91
292–303	Exponential and Logarithmic Functions	3.1, 3.2, 3.3	85–87
309–319	Exponential and Logarithmic Functions	2.1, 6.1	82–84, 92–93
328–331	Sequences and Series	1.1, 1.2, 1.3, 1.4	110–113
339–344	Sequences and Series	4.1	119–121
356–359	Sequences and Series	4.2	119–121
402–408	Permutations and Combinations	1.1	96–97
421–426	Sequences and Series	3.1, 3.3	116–118
477–498	Statistics	2	62–64
503–512	Permutations and Combinations	2.1	98–101
513–516	Permutations and Combinations	3.1	102–104
517–519	Permutations and Combinations	4.1, 4.2	105–107

MATH 30
REVERSE TEXTUAL CORRELATION
NELSON
MATHEMATICS: PRINCIPLES AND PROCESSES

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96–103	Exponential and Logarithmic Functions	2.1	82–84
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115–116	Exponential and Logarithmic Functions	4.1	88–90
119–128	Exponential and Logarithmic Functions	4.2	88–90
130–131	Exponential and Logarithmic Functions	5.1	91
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145–153	Polynomial Functions	5.1, 5.2	37–38
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216–220	Trigonometric and Circular Functions	2.1	44–46
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230–234	Trigonometric and Circular Functions	4.1	51–55
336	Quadratic Relations	1.1	Supplement 4–6
337 (in 9.19 #1)			
349–353	Sequences and Series	1.1, 1.2, 1.3, 1.4	110–113
353–358	Sequences and Series	3.1	116–118
259–363	Sequences and Series	4.1	119–121
364–366	Sequences and Series	2.1	114–115
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372–376	Sequences and Series	4.2	119–121
402–405	Permutations and Combinations	1	96–97
407–410	Permutations and Combinations	2.1	98–101
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MATH 30
REVERSE TEXTUAL CORRELATION
HBJ HOLT PUBLISHING
HOLTMATH 12

Page	Unit	Learner Expectation Number	TRM Page
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63–76	Polynomial Functions	2.1	31–34
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92–93	Polynomial Functions	4.1, 4.2	36
98–103	Polynomial Functions	5.1, 5.2	37–38
122–128	Exponential and Logarithmic Functions	1.1, 1.2, 1.3	80–81
125–128	Exponential and Logarithmic Functions	2.1	82–84
129–132	Exponential and Logarithmic Functions	3.1, 3.2, 3.3	85–87
133–141	Exponential and Logarithmic Functions	4.1, 4.2	88–90
137–139	Exponential and Logarithmic Functions	5.1	91
142–144	Exponential and Logarithmic Functions	6.1	92–93
153	Quadratic Relations	1.1	Supplement 4–6
287–288	Trigonometric and Circular Functions	1	41–43
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296–307	Trigonometric and Circular Functions	3.1	49–50
316–319	Trigonometric and Circular Functions	2.1	44–46
320–324	Trigonometric and Circular Functions	2.2	47–48
327–329	Trigonometric and Circular Functions	4.1, 4.2	51–55
349–352	Permutations and Combinations	1.1	96–97
353–357	Permutations and Combinations	2.1	98–101
358–360	Permutations and Combinations	3.1	102–104
361–362	Permutations and Combinations	4.1	105–107
363–371	Permutations and Combinations	2.1, 3.1	98–104
372	Permutations and Combinations	4.2	105–107
393–403	Statistics	2.1, 2.2, 2.3	
418–420	Sequences and Series	1.1, 1.2, 1.3, 1.4	110–113
421–426	Sequences and Series	3.1, 3.2	116–118
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MATH 30
REVERSE CORRELATION
ADDISON WESLEY 12 NATIONAL EDITION

Topic	Concept #	Page #	B.C. Edition	National Edition
Polynomial Functions	1	30	M12: pp. 42–48	AW12: pp. 42–48
Polynomial Functions	2	34	M12: pp. 42–45, 63–76	AW12: pp. 42–45, 64–78
Polynomial Functions	3	35	M12: pp.	AW12: pp. 74–78
Polynomial Functions	4	36	M12: pp. 49–53	AW12: pp. 49–53
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Trigonometric and Circular Functions	1	43	M12: pp. 156–170	AW12: pp. 198–215
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Trigonometric and Circular Functions	2(cont)	48	M12: pp. 236–240	AW12: pp. 274–288
Trigonometric and Circular Functions	3	50	M12: pp. 178–225	AW12: pp. 261–263
Trigonometric and Circular Functions	4	55	M12: pp. 251–255	AW12: pp. 289–293
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Statistics	2	64	AW: 477–498	AW12: pp. 437–449, 461–475
Statistics	3	70		AW12: pp. 152–192
Exponential and Logarithmic Functions	1	81	AWM12: pp. 262–265, 274–278	AW12: pp. 300–303, 314–318
Exponential and Logarithmic Functions	2	84	M12: pp. 262–265, 309–319	AW12: pp. 300–303, 351–361
Exponential and Logarithmic Functions	3	87	M12: pp. 279–281, 292–303	AW12: pp. 321–323, 334–345
Exponential and Logarithmic Functions	4	90	M12: pp. 282–291, 304–308	AW12: pp. 324–333, 346–350
Exponential and Logarithmic Functions	5	91	M12: pp. 279–281	AW12: pp. 321–323
Exponential and Logarithmic Functions	6	93	M12: pp. 309–319	AW12: pp. 351–361
Permutations and Combinations	1	97	AW30: pp. 500–502	AW12: pp. 484–486
Permutations and Combinations	2	101	AW30: pp. 503–512	AW12: pp. 487–496
Permutations and Combinations	3	104	AW30: pp. 513–516	AW12: pp. 497–505
Permutations and Combinations	4	107	AW30: pp. 517–519	AW12: pp. 506–513
Sequences and Series	1	113	AW: pp. 328–331	AW12: pp. 370–373, 387
Sequences and Series	2	115	AW: pp.	AW12: pp. 410–413
Sequences and Series	3	118	Adison Wesley 12: pp.	AW12: pp. 374–378, 393–397
Sequences and Series	4	121	AW30: pp. 339–344, 356–359	AW12: pp. 381–386, 398–401

QUADRATIC RELATIONS
SUGGESTED PROGRAM EMPHASIS – 10%

Concept	Skills	Problem Solving/ Technology
Students will be expected to demonstrate an understanding of the physical properties of the conic sections with respect to the intersection of a plane and a cone.	Students will be expected to describe the conic section formed by the intersection of a plane and a cone.	Students will be expected to identify the point at which each of the conics becomes degenerate.
Students will be expected to demonstrate an understanding of the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ as the algebraic representation of any conic.	Students will be expected to describe the conics that would be generated by various combinations of values for the numerical coefficients.	Students will be expected to investigate and describe the effects of the numerical coefficients on the graphs of quadratic relations, using calculators or computers.
Students will be expected to demonstrate an understanding of the effects of the numerical coefficients in the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where $B = 0$ on the curves of the resulting conics.	Students will be expected to analyze the graphs of ellipse parabolas and hyperbolas, given their equations.	Students will be expected to use calculators or computers to draw the graphs of ellipse, parabolas and hyperbolas. Students will be expected to recognize which conditions are required for an ellipse to become a circle. Students will be expected to investigate and describe the effects of the numerical coefficients on the orientation, size and shape of the graph.
Students will be expected to demonstrate an understanding that a locus is a system of points that satisfies a given condition.	Students will be expected to recognize that each conic can be described as a locus of points.	Students will be expected to use the locus definition to verify the equations that describe the conics. Students will be expected to solve problems that involve analyzing and determining the characteristics of a body that follows a conical path. Students will be expected to solve problems that involve analyzing and determining the characteristics of a conical surface.
Students will be expected to demonstrate an understanding that any conic can be described as the locus of points, such that the ratio of the distance between any point and a fixed point to the distance between the same point and a fixed line is a constant. This ratio is called eccentricity.		

GENERAL COMMENTS

This approach to the teaching of quadratic relations will be very different for those teachers who are familiar with the pre-September 1991 practice. This new approach follows closely the model for student involvement in the collecting of data, the making of conjectures and the drawing of conclusions that is suggested in Professional Standards for the Teaching of Mathematics, NCTM, 1991. Students will collect and record data, form conjectures and test these conjectures to gain an understanding of quadratic relations. Students will construct their knowledge of the quadratic relations through experimentation and exploration, using a laboratory approach.

As the emphasis is on experimentation and observation, it will be difficult to meet the objectives of this unit without using the available technology to assist in the graphing of the quadratic relations. It is not the intention that students spend their time on paper-and-pencil methods of graphing. Graphing calculators and/or microcomputers with appropriate software will be required to carry out the proposed experimentation with the graphs of the conics. It is the intent that this technology be used in the evaluation as well as in the instruction of this unit. All learning activities and evaluation strategies are based on the premise that students have access to suitable graphing technology.

This section of the TRM for quadratic relations is more detailed than for other units as there are very few resources available in the prescribed textbooks. The activities and suggested approaches cover all concepts to the depth that is required. The activities are developed assuming that the concepts are taught in the order that they are listed in the new course of studies.

Student activities are presented separately so that they may be duplicated for student use. The activities would guide the students in the exploration of the concept. The **Process/ Problem Solving** section includes the rationale for each activity and some possible student outcomes.

In addition to traditional testing, teachers should also use projects, written work, oral presentations, open-ended examinations or a combination of the above. The emphasis should be on graphing and the forming of conjectures from this graphing with less emphasis on algebraic calculations. Evaluation and instructional strategies should reflect this goal.

"Students should have the opportunity to visualize and work with three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. Physical models and other real-world objects should be used to provide a strong base for the development of students' geometric intuition so that they can draw on these experiences in their work with

abstract ideas."[1] The activities are developed with the belief that students construct their own knowledge through experience and that learning is developmental, moving from concrete experiences through visual experiences to the formation of abstract ideas. The activities also allow students to make the connections between the various levels.

1. *Curriculum and Evaluation Standards for School Mathematics*, p. 157, National Council of Teachers of Mathematics, 1989.

CONCEPT 1

Students will be expected to demonstrate an understanding of the physical properties of the conic sections with respect to the intersection of a plane and a cone.

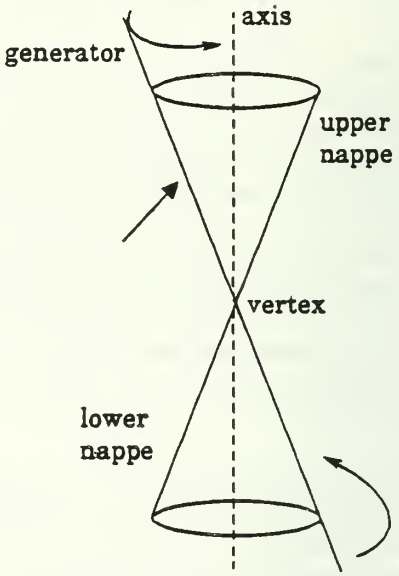
Students will be expected to:

1.1 describe the conic section formed by the intersection of a plane and a cone

1.1.1 identify the point at which each of the conics becomes degenerate

COMMENTS

The Greek mathematician Apollonius, in about 200 B.C., was the first to study the curves produced by the intersection of a plane and a right circular cone with two nappes. These nappes extend indefinitely in both directions. The nappes can be produced by rotating an oblique line (called a generator) about an axis. The intersection of a plane and a double napped cone produces a conic section.



Doubled Napped Cone

PROCESS/PROBLEM-SOLVING CONTEXT

The student activities page includes activities that can be used in the study of this concept. The purpose of these activities is to allow students to discover the basic shapes of the conic sections then later to explore and expand their understanding through writing. The activities progress from the concrete through the pictorial to the abstract.

In **Activity 1.1** and **Activity 1.2** many students will find three basic shapes. They will be unable to distinguish a parabola and an hyperbola but should recognize a circle and ellipse. Some students will distinguish between 'narrow' and 'wide' versions of the same type of conic. In Activity 1.1 the conditions that produce each conic should involve the angle of the flashlight to the wall, whereas Activity 1.2's conditions should refer to the angle of the cut made in the cone. Activity 1.2 could also be done with conical drinking cups or models of cones built from plasticine.

Some students doing **Activity 1.3** will realize that there are actually four different types of shapes. The only difference that students may see between hyperbolas and parabolas is the angle of the plane relative to the axis of the cone that produces each.

After students complete Activity 1.3 they may wonder why there are four shapes. By completing **Activity 1.4** they should realize the difference between a parabola and a hyperbola. Much of the information required to complete this activity is available to students in the prescribed textbooks.

When evaluating student responses to **Activities 1.5** and **1.6** the information below would represent a complete understanding of the concepts involved.

> If the cutting plane cuts through both nappes, the curve of the intersection is called a hyperbola. When the plane passes through the vertex of the cone, the hyperbola degenerates into a pair of intersecting lines.
>
> If the cutting plane is parallel to the generator and does not pass through the vertex, the resulting curve is a parabola. When the plane passes through the vertex of the cone and is parallel to the generator the parabola degenerates into a single line.
>
> If the cutting plane cuts through one nappe only and is not parallel to the generator, the curve of intersection is called an ellipse. A circle is formed when the plane is perpendicular to the axis of the cone and is therefore, the limiting case of the ellipse. When the plane cuts the cone through the vertex, the ellipse or circle degenerates into a point.

STUDENT ACTIVITIES

Activity 1.1: In a darkened room turn on a flashlight and describe the shape of the beam. Describe the different shapes that are produced when the beam is shone against a wall. By placing a piece of paper on the wall it is possible to trace the shapes.

Activity 1.2: Cut a styrofoam cone with a sharp knife. Describe the shape of the various surfaces that result. What conditions produce each shape?

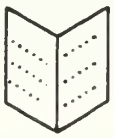
Activity 1.3: Take apart a teacher demonstration cone and trace the curves that result. Describe the conditions that produce each shape.

Activity 1.4: Draw a double napped cone and show the positions of the intersecting planes that produce each of the different conics.

Activity 1.5: Write a paragraph summarizing the conditions that produce each of the conics.

Activity 1.6: When a plane intersects a double napped cone the 'curve' that is produced may **not** be a circle, parabola, ellipse or hyperbola. The points and lines that result are called degenerate conics. Describe the conditions that produce these degenerate conics and the shape of these degenerates. There is more than one degenerate conic.

RESOURCE CORRELATION



HM12: p. 153
MPP12: p. 336, p. 337 exercise 9.19 #1
M12: p. 90–94, p. 94 exercise 3.2 #4

Additional Resources

Videotape

For All Practical Purposes, The Stand-up Conic (available from ACCESS: Catalog Number VC289819).
This tape is a good introduction to conics but covers topics that are outside the requirements of the curriculum. The video gives examples of the usage of conic sections, such as wing design on planes, telescopic lenses, suspension bridges and headlights.

Articles

1. Atkinson, David. *Spheres in a Cone*, or *Proving the Conic Sections*, The Mathematics Teacher, March 1987, pp. 182–187.
   This article establishes the connection between a plane cutting a cone and the locus definitions of the conic sections.

2. Kimberling, Clark. *Conics*, The Mathematics Teacher, May 1984, pp. 363–367.
   This article shows how the study of conics can be enhanced by using computers.

CONCEPT 2

Students will be expected to demonstrate an understanding of the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ as the algebraic representation of any conic.

Students will be expected to:

2.1 describe the conics that would be generated by various combinations of values for the numerical coefficients

2.1.1 investigate and describe the effects of the numerical coefficients on the graphs of quadratic relations, using calculators or computers.

COMMENTS

This approach to the study of conic sections is new and is radically different from the approach used in Mathematics 30 pre-September 1991. The emphasis is now on mathematical ideas generated by investigation and discovery and not on mathematical calculations. This approach does not follow the more traditional calculation of vertices, directrix, foci, major and minor axis but instead focuses on the investigation of the locus by considering the coefficients A, B, C, D, E and F from the general equation. It will provide experiences that allow the student to understand what conditions are necessary for the various conics.

As the prescribed textbooks tend to follow the more traditional approach, other resources have to be consulted in order to meet the objectives of this concept.

Students will discover through open investigation and exploration some basic principles regarding the locus of the general quadratic relation. The emphasis is on student investigation and exploration and away from specific arithmetic calculations. It will be difficult to meet the objectives of this unit without creating the classroom conditions necessary for active student investigation and participation.

As the approach is new and is based on student investigation, teachers have to give some thought to strategies that allow for the evaluation of this concept while keeping within the objectives of open investigation, using the available technology.

In Mathematics 20, students have studied the parabola and have been exposed to the graphing techniques using computer technology that is useful in the investigation of this locus. This concept expands this investigation process to include all conics generated by the general formula. Students are expected to explore and investigate, using the available technology.

PROCESS/PROBLEM-SOLVING CONTEXT

Activity 2.1 is useful for students who have had no experience with graphing technology. It allows practice in entering graphs in terms of the coefficients A, B, C, D, E and F. The results could be discussed to ensure that all have arrived at the same sketch. For students with some experience with graphing technology, this activity could be left out.

Activity 2.2 continues with the study of this concept by having the students suggest some values for A, B, C, D, E and F and then graphing them to see what the locus looks like. Enough time should be spent graphing so that the students are comfortable with the task of graphing a general conic, using the available technology and describing the locus. The students should be encouraged to list the coefficients, equations and the resulting graphs in chart form for easy reference.

Activity 2.3 initiates the actual investigation of this concept. To get students thinking of possibilities, the teacher could have the students graph some equations that have been preselected. One of the equations from **Activity 2.1** has been picked and one coefficient has been changed. When these have been graphed and recorded, the discussion could be directed to the similarities and differences that are apparent and how these could relate to the equations. Note that the conjectures that the students make are based on limited examples and should become more refined as the investigation proceeds.

This process of directed study should give the students some insight into the investigative process for conics and give them some ideas that they would like to try on their own.

While considerable time should be set aside for open investigation of the locus of points, some students may find it instructive if the investigation is guided by WHAT IF questions provided by the teacher.

In Activity 2.9: Students investigate the effect of coefficient B (note that the effect of B is complex and could result in some interesting class or group discussions).

Activities 2.10–2.13: Students are given a graph and asked to find coefficients that would produce this shape. These questions do not have unique solutions.

Activity 2.14–2.19: Ask questions that could be given for investigative purposes.

A possible solution for Activity 2.14 has been provided on pages 9–13.

In the evaluation of this concept, students could be given the coefficients and asked to provide a graph, or students could be given a graph and asked to provide values of the coefficients that would produce the graph. In both cases the students should be able to use the graphing technology. Another form of evaluation may involve a study or project where a student would present his investigations and draw conclusions from these results.

SUMMARY FOR ACTIVITY 2.14

Problem: What shapes are possible when A and C are equal and B is allowed to change?

Collection of data

Since we are testing A, B and C, I let D and E both equal 0. D and E will influence the shape of the curve (see Activity 2) but will not change the type of curve.

I let A and C equal 1. If you increase the values of A and C the same shapes result but the curves are smaller.

I tried 12 graphs to arrive at my conclusions.

Conclusions

By changing the coefficients you can generate a point, a line, two parallel lines, a hyperbola and an ellipse.

If $Ax^2 + Bxy + Cy^2$ forms a perfect trinomial square then a line or lines will result. If F is 0 then only one line will result. If F is any other negative number, two lines will result with the x and y intercepts equal to the square root of F. If F is positive no curve will result.

If the absolute value of B is less than two times the square root of A times the square root of C, then the curve will be an ellipse.

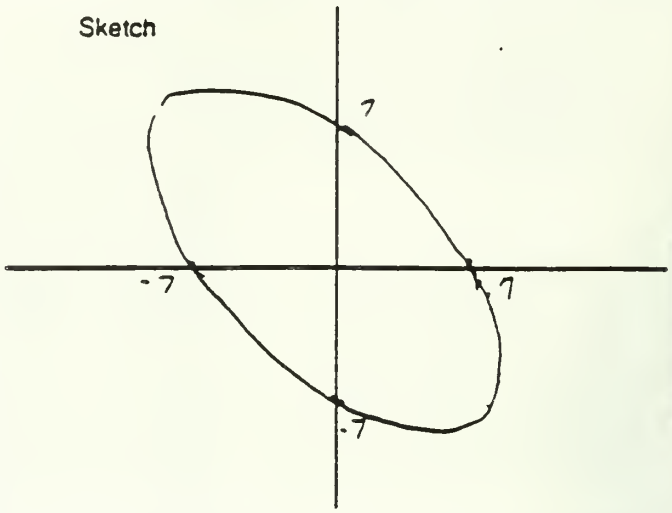
If the absolute value of B is greater than two times the square root of A times the square root of C, then the curve will be a hyperbola.

Activity:

A= 1 B= 1 C= 1 D= 0 E= 0 F= -49

Equation: $x^2 + xy + y^2 - 49 = 0$

Sketch



Observations and Conjectures

To test A, B, C let D ≠ E = 0

Shape is an ellipse

x + y intercepts are equal

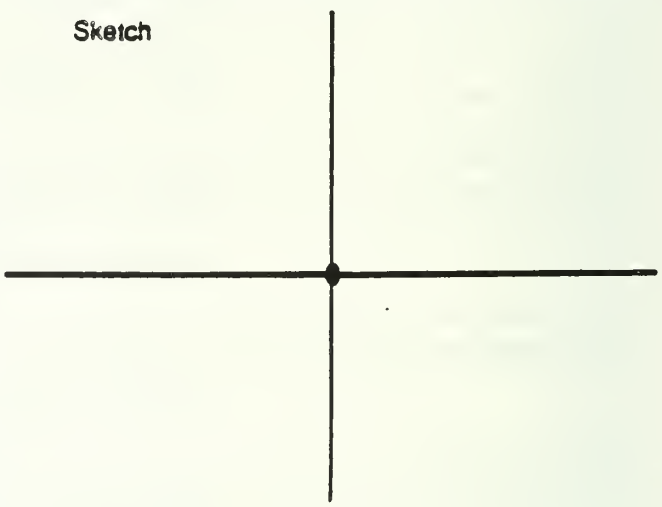
Shape is rotated off axis

Centre is (0,0)

A= 1 B= 1 C= 1 D= 0 E= 0 F= 0

Equation: $x^2 + xy + y^2 = 0$

Sketch



Observations and Conjectures

Point

when $x = 0$, $y^2 = 0$

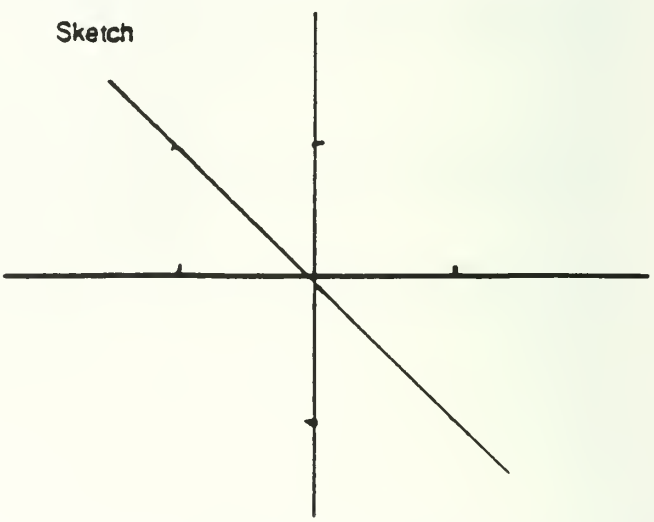
when $y = 0$ $x^2 = 0$

Hard to see on the graph

A= 1 B= 2 C= 1 D= 0 E= 0 F= 0

Equation: $x^2 + 2xy + y^2 = 0$

Sketch



Observations and Conjectures

One line

$x^2 + 2xy + y^2$ is a perfect square

Activity:

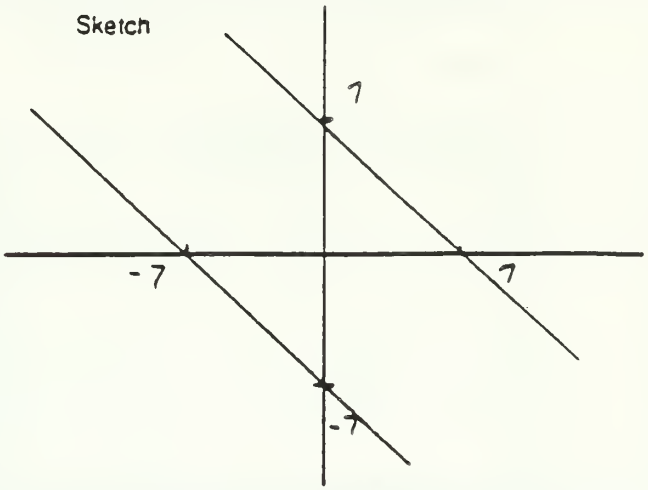
A= 1 B= 2 C= 1 D= 0 E= 0 F= −49

Equation: $x^2+2xy+y^2-49=0$

Observations and Conjectures

Two parallel lines
x + y intercepts are
square root of |F|

Sketch



A= 1 B= 3 C= 1 D= 0 E= 0 F= −49

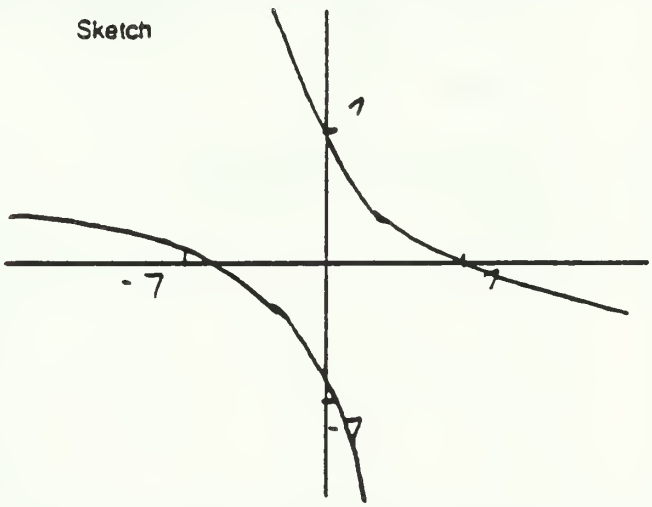
Equation: $x^2+3xy+y^2-49=0$

Observations and Conjectures

Shape is a hyperbola
x + y intercepts are
equal to $\sqrt{F}$

NEXT: Try B > 3

Sketch



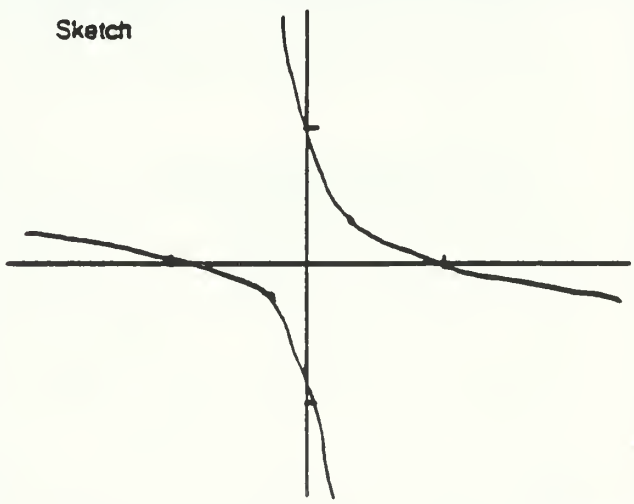
A= 1 B= 4 C= 1 D= 0 E= 0 F= −49

Equation: $x^2+4xy+y^2-49=0$

Observations and Conjectures

Shape is a hyperbola
x + y intercepts are equal
To $\sqrt{F}$
Curve is approaching the
x + y axis
Try a larger B

Sketch



Activity:

A= 1 B= 8 C= 1 D= 0 E= 0 F= -49

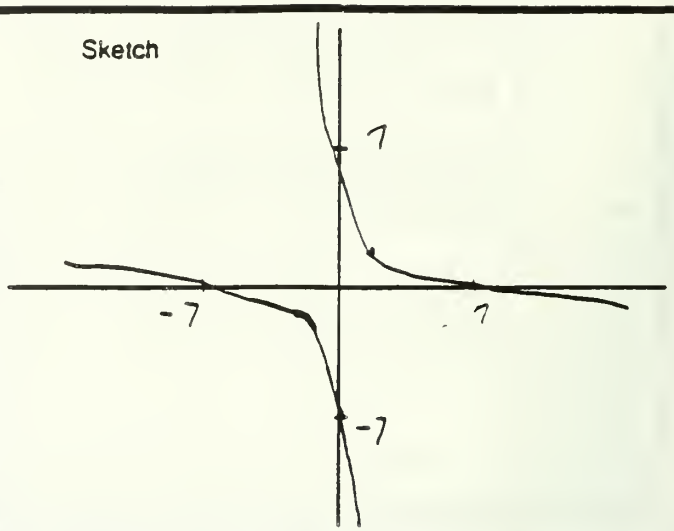
Equation: $x^2 + 8xy + y^2 - 49 = 0$

Observations and Conjectures

Shape is a hyperbola
x + y intercepts same
Closer to x + y axis

Try a large B

Sketch



A= 1 B= 16 C= 1 D= 0 E= 0 F= -49

Equation: $x^2 + 16xy + y^2 - 49 = 0$

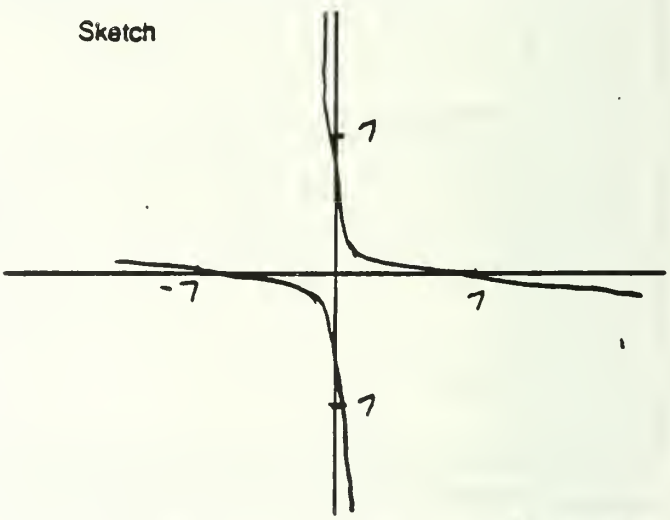
Observations and Conjectures

Shape is a hyperbola
x + y intercepts are the same

As B increases the curve approaches a limit. It will approach the x + y axis

Try a negative B

Sketch



A= 1 B= -1 C= 1 D= 0 E= 0 F= -49

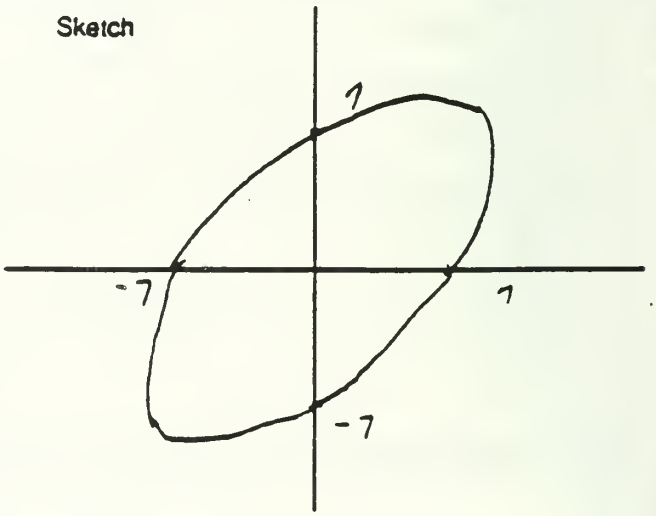
Equation:

Observations and Conjectures

Shape is an ellipse
rotated clockwise
x + y intercepts are equal

negitive values of B change the diretion of rotation

Sketch



Activity:

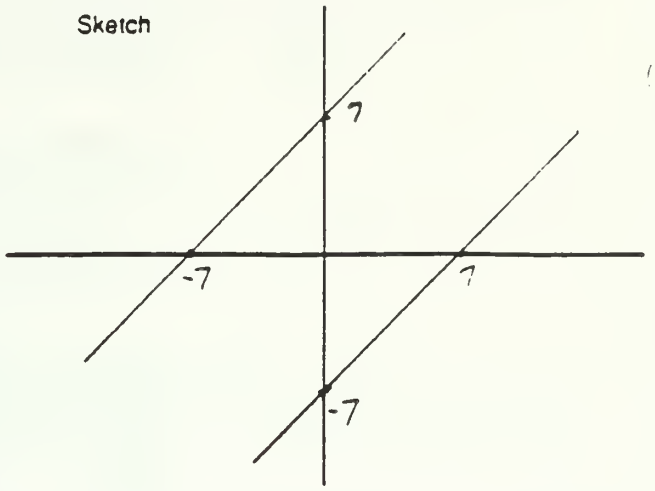
A= 1 B= -2 C= 1 D= 0 E= 0 F= -49

Equation: $x^2 - 2xy + y^2 - 49 = 0$

Observations and Conjectures

Parallel lines
positive slope

Sketch



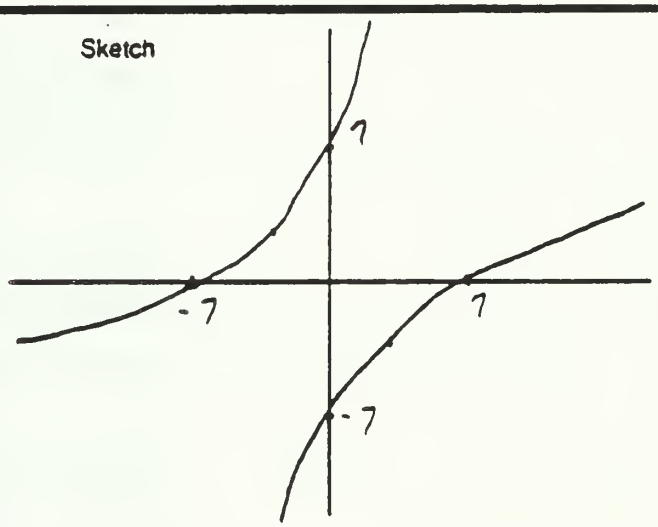
A= 1 B= -3 C= 1 D= 0 E= 0 F= -49

Equation: $x^2 - 3xy + y^2 - 49 = 0$

Observations and Conjectures

Hyperbola
x+y intercepts are equal
rotated clockwise

Sketch



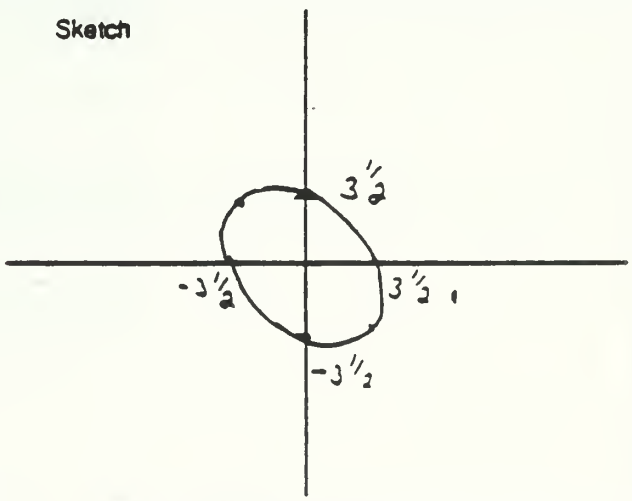
A= 4 B= 4 C= 4 D= 0 E= 0 F= -49

Equation:

Observations and Conjectures

Some shape, only smaller
x+y intercepts equal
intercepts equal $\sqrt{49/4}$

Sketch



STUDENT ACTIVITIES

Activity 2.1: Using the available graphing technology, graph the following relations. Record your results on the data sheets provided for future reference. What types of curves are created by each of these relations? How are these different curves related to the coefficients?

a. $x^2+y^2+4x+8y-40=0$
b. $4x^2+y^2+4x+8y-3=0$
c. $x^2-y^2+4x+8y-16=0$
d. $x^2+4x+6y-1=0$
e. $x^2+xy+y^2+4x+8y-16=0$
f. $2x^2+xy+2x-25=0$

Activity 2.2: One of the equations from Activity 2.1 (b) has been picked and coefficient D has been systematically changed. Determine the effect of changing D on the graph. Record the results on the data sheet and summarize the effect of this change.

a. $4x^2+y^2+8x+8y-3=0$
b. $4x^2+y^2+4x+8y-3=0$
c. $4x^2+y^2+16x+8y-3=0$
d. $4x^2+y^2-8x+8y-3=0$
e. $4x^2+y^2-16x+8y-3=0$
f. $4x^2+y^2-4x+8y-3=0$

Activity 2.3: Start with the equation $x^2+xy+Cy^2+4x+2y-25=0$. What effect does changing the coefficient C have on the curve?

Activity 2.4: What shapes are possible given the equation: $Ax^2+4x-2y-20=0$?

Activity 2.5: Pick another equation from Activity 2.1 and pick a coefficient to experiment with. What is the effect of changing this coefficient on the curve?

Activity 2.6: Investigate the shape of the curve when A and C are equal and $B=0$. What types of shapes are possible?

Activity 2.7: Investigate the shape of the curve when A and C are not equal and $B=0$. What types of shapes are possible?

Activity 2.8: Summarize the types of shapes that are possible if B is set to zero?

Activity 2.9: Pick one of the shapes from Activity 2.7. How is each of the shape changed when B is not equal to zero?

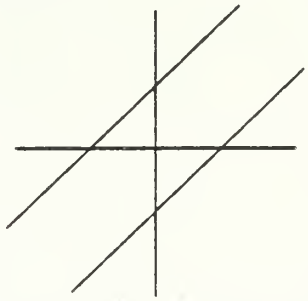
Activity 2.10: What combination of coefficients give this locus?

A= B= C= D= E= F=

Equation:

Observations and Conjectures

Sketch



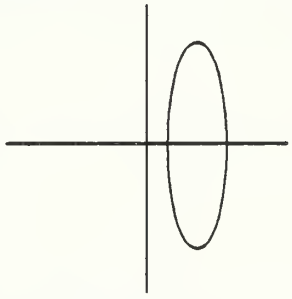
Activity 2.11: What combination of coefficients determine this curve?

A= B= C= D= E= F=

Equation:

Observations and Conjectures

Sketch



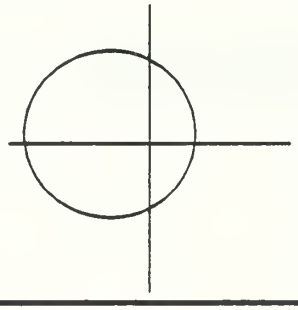
Activity 2.12: What combination of coefficients determine this curve?

A= B= C= D= E= F=

Equation:

Observations and Conjectures

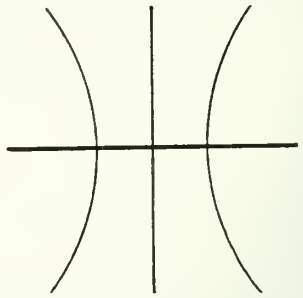
Sketch



Activity 2.13: What combination of coefficients determines this curve?

A= B= C= D= E= F=

Sketch



Equation:

Observations and Conjectures

Activity 2.14: What shapes are possible when A and C are equal and B is allowed to change?

Activity 2.15: Investigate the relationship between the type of curve and the value of B^2-4AC.

Activity 2.16: What shapes are possible given:
$Ax^2+Cy^2+4x+4y-40=0$?

Activity 2.17: Investigate the effect of B on the graph of an ellipse.

Activity 2.18: Investigate $3x^2+4y^2+6x+8y+F=0$ to determine what range of values for F will result in no graph being possible.

Activity 2.19: Devise a problem that follows the format "What happens if . . .?" and test your conjectures using the graphing technology.

TECHNOLOGY INTEGRATION

Students should have access to graphing calculators and/or computers with appropriate graphing software.

Graphing calculators and computers now provide students with the opportunity to draw graphs quickly, make conjectures, draw new graphs to check the conjectures and to draw conclusions.

For graphing calculators the following equations will need to be entered as all graphs need to be in the form y= .

$$y_1=\frac{-1(Bx+E)+\sqrt{((Bx+E)^2-4C(Ax^2+Dx+F))}}{2C}$$

$$y_2=\frac{-1(Bx+E)-\sqrt{((Bx+E)^2-4C(Ax^2+Dx+F))}}{2C}$$

A complete graph can now be obtained by giving A, B, C, D, E and F values.

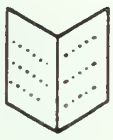
The availability of graphing programs reduces the need for calculation of foci, directrix and vertices as historically these points were used to obtain a quick sketch of the graph.

For a listing of programs that will graph the general quadratic relation, see the appendix under the following headings.

Graphing Calculators:
TI-81 p. 34
Casio 7700G p. 35

Personal Computers:
MSDOS p. 36
Apple II p. 38

RESOURCE CORRELATION



Computer Graphing Experiments 3
Conic Sections
Addison-Wesley Publishing Company

Master Grapher and 3D Grapher
Addison-Wesley Publishing Company

IBM Tool Kit

Zap a Graph
Brain Waves
RR#1 Fitzroy Harbour, Ontario
K0A 1X0

CONCEPT 3

Students will be expected to demonstrate an understanding of the effects of the numerical coefficients in the general quadrants relation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ where $B=0$ on the curves of the resulting conics.

Students will be expected to:

3.1 analyze the graphs of ellipse parabolas and hyperbolas given their equations
- 3.1.1 use calculators or computers to draw the graphs of ellipse, parabolas and hyperbolas
- 3.1.2 recognize which conditions are required for an ellipse to become a circle
- 3.1.3 investigate and describe the effects of the numerical coefficient on the orientation, size and shape of the graph.

COMMENTS

In Concept 2 students have investigated the effects of varying the coefficient 'B' on the shape of the graphs of various conics. It was found that '$B=0$' usually produces a graph that has a vertical and horizontal axis of symmetry.

There are a number of suggested investigations or problems on the student activities page.

Activities 3.1 through 3.7 could be done as group assignments. The key to these investigations is for students to begin with several sets of coefficients and graphs. By looking for a pattern a conjecture can be made which should be checked with another set of coefficients. Through a series of small steps a complex question can be answered.

Students do not need to complete all of the **Activities 3.8 through 3.17**. These are examples of the investigations that students can do. One method of evaluation is to have students suggest and complete their own investigations. In some of these activities students may wish to use their knowledge of the transformation of functions from Mathematics 20. For example, one hyperbola centred at the origin is $16x^2-4y^2-144=0$. By replacing 'x' with '$x-3$' and 'y' with '$y+5$' the centre of the hyperbola is now $(3, -5)$. The resulting equation is $16(x-3)^2-4(y+5)^2-144=0$. By simplifying the resulting expression the student can determine the values for the coefficients 'A' through 'F'.

Activities 3.18 through 3.21 are problems based on analyzing the characteristics of conic sections. Finding the equations of conics that satisfy certain conditions can be done in a number of ways. Algebraic solutions are possible BUT students should be encouraged to make educated guesses which are subsequently checked. The prescribed textbooks have other questions of this type that can be used.

For example, in completing **Activity 3.19** students may use a coordinate system where the origin is at the point where the ball is released or at the base of the cliff. Students may approach this question using concepts developed in the Quadratic Functions unit of Mathematics 20. They may write a general form of an equation like $y = ax^2 + b$. The values of 'a' and 'b' could be found algebraically or through educated guesses.

Another method of solution involves the general form of the parabola $Ax^2 + Dx + Ey + F = 0$. If the origin of the coordinate system is placed at the base of the cliff then the points (0, 30) and (55, 0) must be on the curve. Some students may realize that the vertex of the parabola must be at the point of release of the ball. If this is not the case, then many parabolas could be found that go through the two points. By systematically changing the coefficients a solution can be found.

STUDENT ACTIVITIES

Note: All activities use the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where it is assumed that '$B = 0$'.

Activity 3.1: What restrictions on coefficients of the general quadratic relation result in a parabola?

Activity 3.2: What restrictions produce a ellipse?

Activity 3.3: What restrictions produce a hyperbola?

Activity 3.4: What change in the coefficients causes an ellipse to become a circle?

Activity 3.5: What restrictions on the coefficients produce a parabola which has a vertex at the origin?

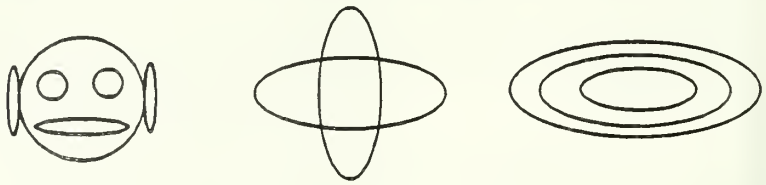
Activity 3.6: If an ellipse, hyperbola or circle are centred at the origin then what restrictions on the coefficients are required?

Activity 3.7: Write a summary of the conclusions that were made in Activities 3.1 through 3.6.

Activity 3.8: Given $x^2 + 4y^2 + Dx + Ey - 16 = 0$, find values of D and E that would move the ellipse to a position with centre in quadrant two.

Activity 3.9: Given $Ax^2 + Cy^2 + 4x + 6y = 0$, how would A and C compare if the longer axis of the ellipse was the horizontal axis?

Activity 3.10: Make up equations, graph them using an overlay procedure, so that they resemble one of the pictures below.



Activity 3.11: Find an equation of an ellipse that has no intercepts and is located in the fourth quadrant.

Activity 3.12: Find an equation of a parabola that has its vertex in the third quadrant and opens upward.

Activity 3.13: Find an equation of a hyperbola with centre in the second quadrant.

Activity 3.14: What combination of coefficients produces a hyperbola that has a horizontal axis of symmetry?

Activity 3.15: If given the equation of a hyperbola then what changes must be made in the coefficients in order to produce a 'new' hyperbola with a wider opening but the same centre.

Activity 3.16: Given the equation of a parabola, what changes must be made in the coefficients to produce a 'new' parabola, with a narrower opening and with the same vertex?

Activity 3.17: Given the equation of a parabola, what changes must be made in the coefficients to produce a 'new' parabola that is the mirror image of the original and with the same vertex?

Activity 3.18: Find an equation of an ellipse that passes through the point (3, 4).

Activity 3.19: A ball was thrown horizontally from the top of a 30 m cliff and followed a parabolic path. The ball landed 55 m from the base of the cliff. Find an equation of the path that the ball followed.

Activity 3.20: During a football game, a kicker is attempting a field goal from 35 yards. The bar on the goal posts is 3.5 yards high. The ball must clear the upraised hands of the defensive linemen who are 7 yards away and be able to reach to a height of 3.5 yards. Assume that the ball lands 7 yards behind the goal posts. Find the equation of the parabolic path of the ball that just clears both the lineman's hands and the bar on the goal posts. Also find the maximum height of the ball in its path. (Note: The game of football uses 'yards' as the unit of length, not the SI units, 'metres'.)

Activity 3.21: A trailer is 3.4 m wide. The entrance to a snow shed in the Roger's Pass is in the shape of a semi-ellipse with a width of 8.0 m and height of 5.0 m.

a. If the trailer is to be pulled through the centre of the entrance then what is the maximum height of the trailer?

b. If there is two-way traffic through the shed then what is the maximum height of the trailer?

CONCEPT 4

Students will be expected to demonstrate an understanding that a locus is a system of points that satisfies a given condition.

Students will be expected to:

4.1 recognize that each conic can be described as a locus of points
- 4.1.1 use the locus definition to verify the equations that describe the conics
- 4.1.2 solve problems that involve analyzing and determining the characteristics of a body that follows a conical path
- 4.1.3 solve problems that involve analyzing and determining the characteristics of a conical surface.

COMMENTS

The locus definitions in these concepts are geometric conditions that describe a set of points. Students will be given activities that draw the graphs of the conics. From these graphs, the students will write a rule or definition of the locus. Subsequent activities lead from the locus definition to the equation of the curve. Once this equation has been obtained the student should be able to verify the rule by picking points on the locus.

PROCESS/PROBLEM-SOLVING CONTEXT

The first three activities are taken from work that the students have done in Mathematics 10 and 20 and are included here to remind the students that the paths of points can be described by some rule that defines all points on the curve.

Activities 4.4, 4.5 and 4.6 use one of the approved text resources and have the students construct a parabola, an ellipse and a hyperbola. The students write a rule that defines the points on the curves.

In **Activity 4.7** the students are asked to rewrite all of the rules in **Activities 4.2 through 4.6** in locus definition form.

The paper folding activities, **Activity 4.8 through 4.11**, are alternative methods for drawing a parabola, an ellipse and a hyperbola. They are included as there is some advantage to considering different constructions and models for the same mathematical relations. Wax paper is required for these activities.

Activities 4.12 through 4.13 use the double concentric circle graph paper to construct an ellipse and a hyperbola. This is another approach to the conic relations. The constant sum and difference are calculated by counting concentric rings on the graph paper. For those not familiar with this graph paper, an example has been included in the appendix.

Activities 4.14 through 4.17 ask the students to draw the curve from the locus definition and then to find the equation that describes this locus. The equations can be calculated by selecting points on the locus and using systems of equations or by a process of educated guessing. Students may not be required to do all of **Activities 4.14 through 4.17**.

Activities 4.19 through 4.21 ask the students to verify the locus definitions. This should be done using the distance formula.

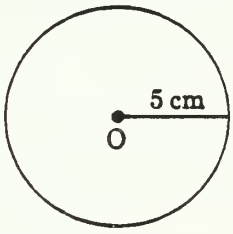
Activities 4.22 through 4.23 introduce problems that involve analyzing and determining the characteristics of conics.

STUDENT ACTIVITIES

Activity 4.1: Choose two points A and B on a sheet of paper. Find all of the points that are equidistant from these two points.

Activity 4.2: Draw the set of points that includes (0, 4) and follow the rule: "the slope between any two points is 3".

Activity 4.3: Given the set of points as shown in the diagram, write a rule that will define all points on the curve.



Activity 4.4: Using a compass and a lined sheet of paper, construct a parabola using the procedures as outlined in the INVESTIGATION section, page 100, *Addison-Wesley Mathematics 12*. Write a rule that will define all points on the curve.

Activity 4.5: Construct an ellipse using the procedures as outlined in the INVESTIGATION section, page 112, *Addison-Wesley Mathematics 12*. Write a rule that will define all points on the curve.

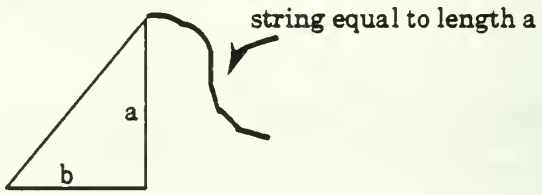
Activity 4.6: Construct a hyperbola using the procedures as outlined in the INVESTIGATION section, page 119, *Addison-Wesley Mathematics 12*. Write a rule that will define all points on the curve.

Activity 4.7: A locus is a path or a set of points that follow a rule. The rule itself is called a locus definition. A locus definition for **Activity 4.1** would be: "the line is the locus of points equidistant from the two fixed points A and B". Rewrite the rules in **Activities 4.2 through 4.6** as locus definitions.

Activity 4.8: Using a blackboard set square, a straightedge, and a piece of string construct a shape by following these steps:

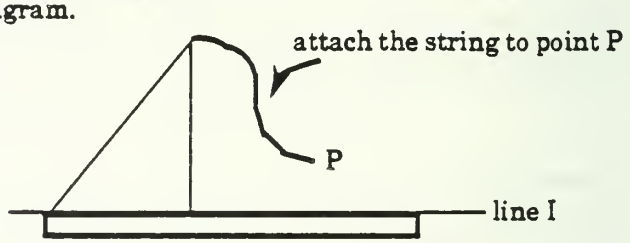
1. Tape the string to the top of the set square.

2. Cut the string to equal the length of the side of the set square.



3. Draw line I and pick point P above the line.

4. Place the straightedge, the square and the string as shown in the diagram.



5. With the pencil holding the string against the edge (a) of the square, draw down the edge (a) of the square. Let the square move freely along the straightedge.

6. Move the square to the other side of P and complete the shape.

What curve results?
Write a locus definition for this set of points.

Activity 4.9: Cut a circle from a piece of waxed paper. Mark a point P inside the circle. Pick a point on the edge of the circle and fold it so that it touches point P. Crease the paper. Repeat these steps with different points on the circumference of the circle until the curve that appears can be described. What conic is created? Why does this procedure produce this curve?

Activity 4.10: Cut a rectangle from a piece of waxed paper. Mark a point P inside the rectangle. Pick a point on the bottom edge of the rectangle and fold it so that it touches point P. Crease the paper. Repeat these steps with different points on the edge of the rectangle until the curve that appears can be described. What conic is created? Why does this procedure produce this curve?

Activity 4.11: Draw a circle on a rectangular piece of waxed paper. Mark a point P outside the circle. Fold the paper so that point P touches a point on the circumference. Crease the paper. Repeat these steps with different points on the circumference of the circle until the curve that appears can be described. What conic is created? Why does this procedure produce this curve?

Activity 4.12: Use double concentric circle graph paper. (See Appendix C.)

1. Label the two centres A and B.
2. Select a number (keep the number less than 20).
3. Find a point P such that the distance PA + PB adds to the selected number.
4. Find other points that satisfy this condition.
5. Continue this process until you can draw the complete curve.

What shape is this curve?
Why does this procedure produce this curve?

Activity 4.13: Use double concentric circle graph paper. (See Appendix C.)

1. Label the two centres A and B.
2. Select a number.
3. Find a point P such that the distance $|PA - PB|$ is equal to the number.
4. Find other points that satisfy this condition.
5. Continue this process until you can draw the complete curve.

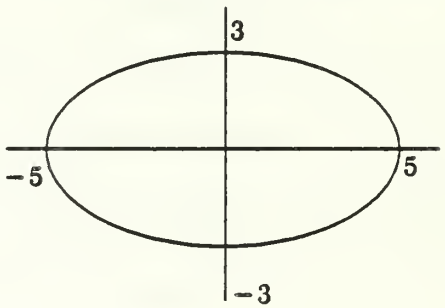
What shape is this curve?
Why does this procedure produce this curve?

Activity 4.14: Construct a parabola on graph paper, using the procedures from **Activity 4.4**. Draw a set of axes on this graph paper. Write the equation of this parabola. Write the equation of the directrix. Label the coordinates of the focus.

Activity 4.15: Construct an ellipse on graph paper, using the procedures from **Activity 4.5**. Draw a set of axes on this graph paper. Write the equation of this ellipse. Label the coordinates of the foci.

Activity 4.16: Construct a hyperbola on graph paper, using the procedures from **Activity 4.6**. Draw a set of axes on this graph paper. Write the equation of this hyperbola. Label the coordinates of the foci.

Activity 4.17: Given the following diagram, determine an equation that will produce this ellipse.



Challenge: What is the constant sum?
Give the coordinates of the foci.

Activity 4.18: Given the equation $4x^2+9y^2-36=0$, find the constant sum.

Activity 4.19: Given the equation $x^2-16y-16=0$, with directrix $y=-4$ and focus (0, 4), by selecting two points on the locus, verify the locus definition for this curve.

Activity 4.20: Given the equation $9x^2-16y^2-144=0$, with foci $(-5, 0)$ and $(5, 0)$, by selecting two points on the locus, verify the locus definition for this curve.

Activity 4.21: Given the equation $9x^2+25y^2-225=0$, with foci $(-4, 0)$ and $(4, 0)$, by selecting two points on the locus, verify the locus definition for this curve.

Activity 4.22: Two stakes are driven into the ground 20 m apart and a rope 40 m in length is fastened to the stakes. The rope runs through the ring on a guard dog's collar.

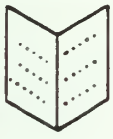
a. What is the equation that describes the perimeter of an area within which the dog can move?

b. A sidewalk is parallel to a line through the stakes and 30 m from this line. What is the maximum length of the rope that you can use if the dog is **not** to reach the sidewalk?

Activity 4.23: An airplane can fly a maximum of 1000 km in one day. Peace River and Athabasca are 300 km apart. The pilot always starts his trip in one city and finishes in the other. He may take side trips during the flight. Within what figure would the pilot be confined to fly?

Activity 4.24: Find the equation of the ellipse when the constant sum is 12, and the foci are $(-3, 0)$ and $(3, 0)$.

RESOURCE CORRELATION M12: pp. 100–112, 119



CONCEPT 5

Students will be expected to demonstrate an understanding that any conic can be described as the locus of points, such that the ratio of the distance between any point and a fixed point to the distance between the same point and a fixed line is a constant. This ratio is called eccentricity.

COMMENTS

The student activities page contains some investigations and problems that are appropriate for the study of this concept. **Activity 5.1** builds on the locus definitions of the previous concept. Teachers may want to read the introduction to this activity now. The results that students should find are summarized in the table below. The ratio referred to in this activity is called the eccentricity. As the eccentricity of the ellipse approaches zero then the ellipse approaches a circle.

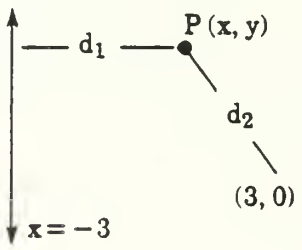
CONIC	ECCENTRICITY
PARABOLA	EQUAL TO ONE
ELLIPSE	BETWEEN 0 AND 1
HYPERBOLA	GREATER THAN 1

Samples of the circle-line graph paper are contained in the appendix of this document. An example of using this graph paper is on the next page. Students may need some assistance initially in using this graph paper.

Activity 5.2 is similar to the investigations done in Concepts 3 and 4. One value is allowed to vary while the others are changed. In this activity the directrix is allowed to move while the focus and eccentricities change. Students should summarize their findings in writing. It may be appropriate to assign different eccentricities to various groups.

In **Activity 5.3** a single directrix and focus are used to draw an ellipse, hyperbola and a parabola. The eccentricity of each can be determined by using the distance formula and points that are on each conic.

Students in **Activity 5.4** are asked to use the locus definition to determine whether or not a point is part of the locus. Also, the locus definition is used to find the equation of the resulting conic. (A solution appears below.) Teachers could also ask students to make up a locus definition of a conic and then to find the equation of the locus.



$$\frac{d_2}{d_1} = \frac{\sqrt{(x-3)^2+y^2}}{x+3}$$

$$\frac{d_2}{d_1} = 2 \text{ (eccentricity is 2)}$$

$$\sqrt{(x-3)^2+y^2} = 2x+6$$

$$x^2-6x+9+y^2 = 4x^2+24x+36$$

$$3x^2-y^2+30x+25 = 0$$

STUDENT ACTIVITIES

Activity 5.1: The locus definition of the parabola involves the distance from a fixed point called a focus and the distance to a fixed line called a directrix. For any point on a parabola the distances are equal.

distance to the focus = distance to the directrix (PARABOLA)

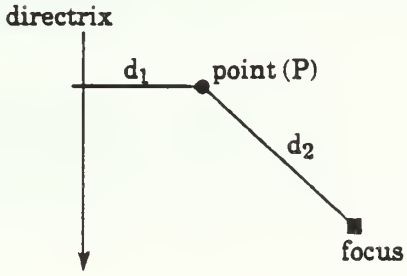
Another way to state this definition is to say that the ratio of the distances is equal to one.

$$\frac{\text{distance to the focus}}{\text{distance to the directrix}} = 1 \text{ (PARABOLA)}$$

What happens if this ratio is not equal to 1? On the circle-line paper provided choose the focus to be at the centre of the concentric circles. Choose any line to be the directrix. Draw the locus of points where the ratio has the following values:

a. 2/3
b. 3/2
c. 3/4
d. 4/3
e. 1

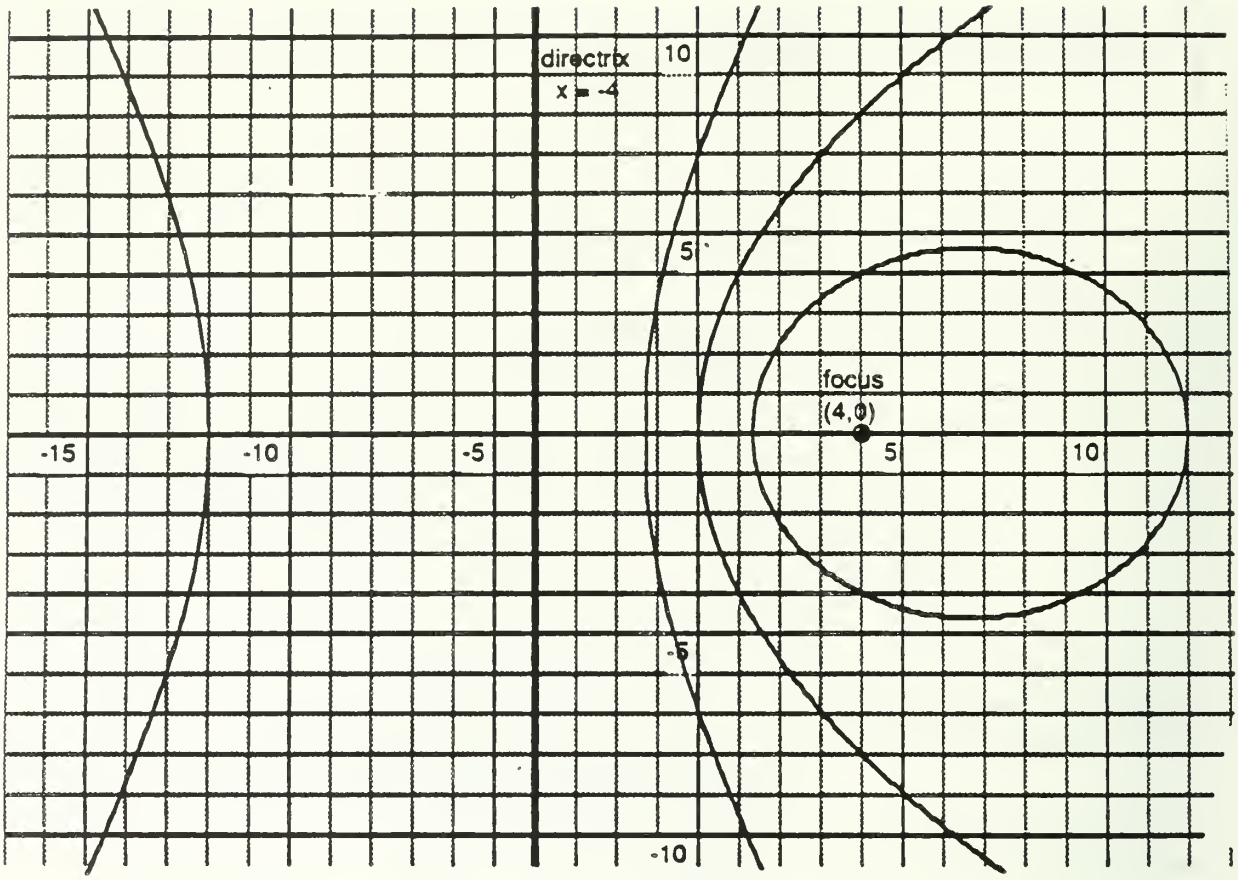
Based on the results, what conjectures can be made? Check these conjectures with other ratios. Summarize the results in several sentences.



The Ratio d_2/d_1 is called the eccentricity.

Activity 5.2: The ratio referred to in **Activity 5.1** is called the eccentricity. Use the circle-line graph paper to determine what happens to the locus of points if the eccentricity is fixed and the directrix changes position. Summarize any findings in a paragraph.

Activity 5.3: Find the eccentricity of each of the conics on the following page.



Activity 5.4: Find three points on the conic with a directrix at $x+3=0$, focus at (3, 0) and an eccentricity of 2. Verify that these points are on the conic by using the distance formula. Find the equation of the conic and sketch the graph.

APPENDICES

Appendix A – Graphing Data Sheet

Appendix B – Sample Calculator and Computer Graphing Programs

Appendix C – Sample Graph Paper

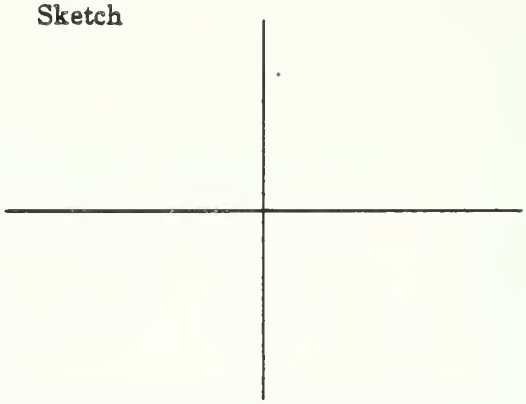
GRAPHING DATA SHEET

Activity _____

A= B= C= D= E= F=
Equation:

Observations and Conjectures

Sketch

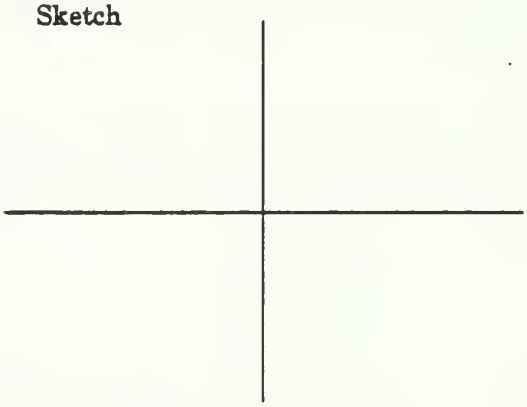


Activity _____

A= B= C= D= E= F=
Equation:

Observations and Conjectures

Sketch

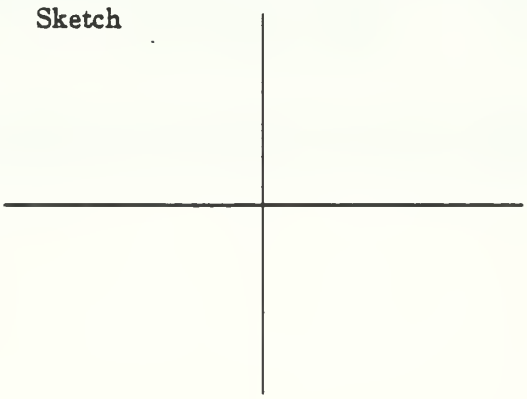


Activity _____

A= B= C= D= E= F=
Equation:

Observations and Conjectures

Sketch



SAMPLE CALCULATOR AND COMPUTER GRAPHING PROGRAM

T1–81 Graphing Calculator

```
Prgm1:CONICS
:Disp "A"
:Input A
:Disp "B"
:Input B
:Disp "C"
:Input C
:Disp "D"
:Input D
:Disp "E"
:Input E
:Disp "F"
:Input F
:If C≠0
:Goto 3
:"(-1*A*X^2-D*X-F)/(B*X+E)"->Y1
:DispGraph
:End
:Lbl 3
:"(-1*(B*X+E)+√((B*X+E)^2-4*C*(A*X^2+D*X+F)))/(2*C)"->Y1
:"(-1*(B*X+E)-√((B*X+E)^2-4*C*(A*X^2+D*X+F)))/(2*C)"->Y2
:DispGraph
:End
```

To enter the program: push PRGM then right arrow to EDIT then ENTER
After each line push ENTER

Symbol	Key Strokes
"	ALPHA then the + key
A	ALPHA then A
B	ALPHA then B
Disp	PRGM then right arrow to I/O, arrow up or down to highlight selection, ENTER
Input	PRGM then right arrow to I/O, arrow up or down to highlight selection, ENTER
≠	2nd, then Math, arrow to selection then ENTER
Y_1	2nd, then VARS, arrow to selection, ENTER
->	STO
If	PRGM, then arrow to If, ENTER
Goto	PRGM, then arrow to Goto, ENTER
Disp Graph	PRGM then right arrow to I/O, arrow up or down to highlight selection, ENTER
END	PRGM, then arrow to END, ENTER
Lbl	PRGM, then ENTER

When you have finished entering the program, push 2nd, then CLEAR
To run the program push PRGM, then ENTER, then ENTER.
ZOOM (Arrow to Square, Enter) This makes the axis square on the screen.

CASIO∫x – 7700G Power Graphic

To start programming: MODE, 2, arrow down to an empty program number, EXE.
To get letters use the ALPHA key.

CONICS
"A="?→A:"B="?→B:"C="?→C exe
"D="?→D:"E="?→E:"F="?→F exe
C=0⇒(−1×A×X×y²−D×X−F)÷(B×X+E)▸
Graph y=((−1×(B×X+E)+√((B×X+E)×y²−4×C×(A×X×y²+D×X+F)))÷(2×C) exe
Graph y=((−1×(B×X+E)−√((B×X+E)×y²−4×C×(A×X×y²+D×X+F)))÷(2×C)▸

When the programming is completed: MODE, 1, AC.

To run this program: SHIFT, PRGM, F3, Program number, EXE.

Comments

To get these symbols and commands

"	ALPHA, F2
=	SHIFT, PRGM, F2, F1
?	SHIFT, PRGM, F4
:	SHIFT, PRGM, F6
⇒	SHIFT, PRGM, F1, F1
Graph y=	Graph key
▸	SHIFT, PRGM, F5
√,→,(,),	are actual keys

RANGE: Adjust the range so that the scale on x is −14.1 to 14.1 and the scale on y is −9.3 to 9.3 (this helps to make a circle look like a circle).

To clear the screen SHIFT, F5, EXE

The following is a program listing for MSDOS computers to graph conics and degenerate conics in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. To make your copy of the program, load GWBASIC, type the listing and save the program using SAVE "GENCONIC". To use the program load GWBASIC, and type LOAD "GENCONIC", followed by RUN.

```
100  KEY OFF : SCREEN 1 : CLS 2 : COLOUR 1, 1
110  XMIN = -10: XMAX = 10: YMIN = -10: YMAX = 10
120  GOSUB 1000 : 'Set up screen
130  LOCATE 5, 1 : INPUT "A = ", A : INPUT "B = ", B : INPUT "C = ", C
140  INPUT "D = ", D : INPUT "E = ", E : INPUT "F = ", F : PRINT
150  DEF FN T(X) = (B*X + E)^2 - 4*C*(A*X^2 + D*X + F)
160  DEF FN F(X) = ( -(B*X + E) + SQR(FN T(X)))/(2*C)
170  DEF FN G(X) = ( -(B*X + E) - SQR(FN T(X)))/(2*C)
180  DEF FN H(X) = ( - A*X^2 - D*X - F)/(B*X + E)
190  IF C = 0 AND B = 0 AND E = 0 GOTO 210
200  GOSUB 3000 : GOTO 220 : 'Draw graph
210  GOSUB 2000 ; 'Vertical line graphs
220  IF REL = 1 THEN PRINT "Not a" : PRINT "relation." : GOTO 280
230  LOCATE 14, 1 : PRINT "New scale" : INPUT "(y/n)";A$
240  IF A$ < > "y" AND A$ < > "Y" GOTO 280
250  LOCATE 15, 1 : INPUT "lower x";XMIN : INPUT "upper x";XMAX
260  INPUT "lower y";YMIN : YMAX = YMIN + XMAX - XMIN
270  GOSUB 1000 : GOTO 190 : 'New screen
280  PRINT : PRINT "New graph" : INPUT "Y/N)";A$
290  IF A$ = "y" OR A$ = "Y" THEN A$ = " " : REL = 0 : CLS 2 : GOTO 120
300  SCREEN 0, 0, 0 : WIDTH 80 : END
1000 '----------Screen set up----------
1010 LOCATE 1,3 : PRINT "Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0"
1020 LOCATE 14,1 : FOR I = 1 TO 6 : PRINT "        " : NEXT I
1030 LOCATE 3,1 : PRINT "Conics":LOCATE 20, 1:PRINT "Scale":PRINT
1040 PRINT "x = "XMIN;" to "XMAX;" " : PRINT "Y = "YMIN;" to "YMAX;" "
1050 VIEW (110, 18) - (310, 190), 2 : WINDOW (XMIN, YMIN) - (XMAX, YMAX)
1060 LINE (XMIN, 0) - (XMAX, 0) : LINE (0, YMIN) - (0, YMAX)
1070 FOR X = XMIN TO XMAX : LINE (X, -.1) - (X, .1) : NEXT X
1080 FOR Y = YMIN TO YMAX : LINE (-.1, Y) - (.1, Y) : NEXT Y
1090 RETURN
2000 '----------Vertical Line Graphs----------
2010 IF A = 0 AND D = 0 THEN REL = 1 : GOTO 2060
2020 IF A = 0 THEN LINE (F/D, YMIN) - (F/D, YMAX) : GOTO 2060
2030 IF D^2 - 4*A*F < 0 THEN REL = 1 : GOTO 2060
2040 X1 = ( - D - SQR(D^2 - 4*A*F))/(2*A) : X2 = ( - D + SQR(D^2 - 4*A*F))/(2*A)
2050 LINE (X1, YMIN) - (X1, YMAX),1 : LINE (X2, YMIN) - X2, YMAX),1
2060 RETURN
3000 '----------Draws Graphs----------
3010 INC = (XMAX - XMIN)/200 : REL = 1
3020 FOR X = XMIN TO XMAX STEP INC
3030 X = INT(X*100 + .5)/100
3040 IF C = 0 AND ABS (B*X + E) < .001 GOTO 3080
3050 IF C = 0 THEN PSET (X, FN H(X)) ,1 : REL = 0 : GOTO 3080
3060 IF FN T(X) < 0 GOTO 3080
3070 PSET (X, FN G(X)),1 : PSET (X, FN F(X)),1 : REL = 0
3080 NEXT X
3090 RETURN
```

The preceeding program can be modified to graph conics using the focus-directrix definition. That is a conic will be drawn if the location of the focus is given, a directrix in the form $Ax+By+C=0$ is provided and the eccentricity is provided.

To make your copy first delete lines 130, 140, 1010 and 1030.

Then include the following lines. When you have typed these lines, save the program using SAVE "FDCONIC".

```
125  LOCATE 4,1 : INPUT "x=", F : INPUT "y=",V
130  PRINT "Directrix":PRINT "Ax+By+C=0"
135  INPUT "A=",A : INPUT" B=",B : INPUT "C=",C
140  PRINT "Eccentricity" : INPUT "e=",E : PRINT
142  A1=E^2×A^2-A^2-B^2 : B1=2×A×B×E^2 :C1=E^2×B^2-A^2-B^2
144  D1=2×A×C×E^2+2×A^2×F+2×B^2×F : E1+2×B×C×E^2+2×A^2×V+2×B^2×V
146  F1=E^2×C^2-A^2×F^2-A^2×V^2-B^2×F^2-B^2×V^2
148  A=A1 : B=B1 : C=C1 : D=D1 : E=E1 : F=F1

1010 LOCATE 1,1 : PRINT "Conics : The Focus-Directrix Definition"
1030 LOCATE 3,1:PRINT "FOCUS F(x,y)":LOCATE 20,1:PRINT "Scale" :PRINT
```

The following is a program listing for Apple computers to graph conics and degenerate conics in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. To make your copy of the program type the listing and save the program using SAVE GENERAL CONIC.

The speed of the program varies inversely as the quality of the graph. The speed can be increased by reducing the number 60 in line 5010.

```
100  CLEAR : DEF FN P(X)+XC+UN×X: DEF FN Q(Y)=YC-UN×Y
110  XL=-6:XH=6:YL=-6:YH=6
120  GOSUB 2000: REM SET SCREEN
130  IF SC < > 1 THEN GOSUB 1000: REM INTERACTION
140  IF SC < > 1 THEN GOSUB 3000: REM DEFINE FUNCTION
150  VTAB 23: PRINT "SCALE X=";XL;" TO ";XH
160  VTAB 23: HTAB 28: PRINT "Y=";YL;" TO ";YH
170  IF B=0 AND C=0 AND E=0 THEN GOSUB 4000: GOTO 190
180  GOSUB 5000: REM DRAW GRAPH
190  IF REL=1 THEN PRINT "NOT A RELATION. ";: GOTO 220
200  SC=0: VTAB 24: INPUT "NEW SCALE (Y/N)?";A$
210  IF A$="Y" THEN A$=" ": GOSUB 6000: REM NEW SCALE
220  IF SC < > 1 THEN VTAB 24: INPUT "NEW GRAPH (Y/N)?";A$
230  IF A$="Y" THEN GOTO 100
240  IF SC=1 THEN GOTO 120
250  END : REM  ------------------------
1000 REM INTERACTION
1010 VTAB 21: PRINT " AX^2+BXY+CY^2+DX+EY+F=0"
1020 VTAB 22: HTAB 1: INPUT "A=";A$:A= VAL (A$)
1030 VTAB 22: HTAB 7: INPUT "B=";B$:B= VAL (B$)
1040 VTAB 22: HTAB 13: INPUT "C=";C$:C= VAL (C$)
1050 VTAB 22: HTAB 19: INPUT "D=";D$:D= VAL (D$)
1060 VTAB 22: HTAB 25: INPUT "E=";E$:E= VAL (E$)
1070 VTAB 22: HTAB 31: INPUT "F=";F$:F= VAL (F$)
1080 RETURN
2000 REM SCREEN SET UP ----------
2005 WI=XH-XL:UN=INT (160 / WI)
2010 XC=140- INT ((XH+XL) / 2×UN)
2020 YC=80+INT ((YH+YL) / 2×UN)
2030 HOME : HGR : HCOLOR=3
2040 FOR I=XL TO XH
2050 HPLOT FN P(I), FN Q(0) TO FN P(I), FN Q(.1)
2060 NEXT I
2070 FOR I=YL TO YH
2080 HPLOT FN P(0), FN Q(I) TO FN P(.1), FN Q(I)
2090 NEXT I
2100 HPLOT FN P(XL), FN Q(0) TO FN P(XH), FN Q(0)
2110 HPLOT FN P(0), FN Q(YL) TO FN P(0), FN Q(YH)
2120 RETURN
3000 REM DEFINES FUNCTION ----------
3010 DEF FN T(X)=(B×X+E)^2-4×C×(A×X^2+D×X+F)
3020 DEF FN F(X)=(-(B×X+E)+SQR (FN T(X))) / (2×C)
3030 DEF FN G(X)=(-(B×X+E)-SQR (FN T(X))) / (2×C)
3040 DEF FN H(X)= (-A×X^2-D×X-F) / (B×X+E)
3050 RETURN
4000 REM VERTICAL LINES ----------
4010 IF A=0 AND D=0 THEN REL=1: GOTO 4060
4015 IF A=0 THEN XX= FN P(F / D)
4020 IF A=0 THEN HPLOT XX, FN Q(YL) TO XX, FN Q(YH): GOTO 4060
4030 DI=D^2-4×A×F: IF DI < 0 THEN REL=0: GOTO 4060
4040 X1=(-D-SQR (DI)) / (2×A):X2+(-D+SQR (DI)) / (2×A)
```

```
4050  HPLOT FN P(X1), FN Q(YL) TO FN P(X1), FN Q(YH)
4055  HPLOT FN P(X2), FN Q(YL) TO FN P(X2), FN Q(YH)
4060  RETURN
5000  REM  DRAWS GRAPH ----------
5010  INC=WI/60:REL=1: IF INC < .1 THEN INC=.1
5020  FOR I=XL TO XH STEP INC
5022  I=INT (I×10+.5) / 10
5025  IF C=0 AND ABS (B×I×E) < 0.001 THEN GOTO 5080
5030  X=FN P(I): IF X < 60 OR X > 220 THEN 5080
5050  IF C=0 THEN J=FN H(I): GOTO 5065
5055  IF FN T(I) < 0 THEN 5080
5060  J=FN G(I):N=1
5065  Y=FN Q(J): IF Y < 0 OR Y > 159 THEN 5073
5070  HPLOT X, Y
5073  IF N=1 THEN J=FN F(I):N=0: GOTO 5065
5075  REL=0
5080  NEXT I
5090  RETURN
6000  REM  NEW SCALE ----------
6010  SC=1: HOME
6020  VTAB 21: INPUT "LOWER X";XL$:XL=VAL (XL$)
6030  IF XL > 0 THEN GOTO 6020
6040  VTAB 22: INPUT "UPPER X ";XH$:XH= VAL (XH$)
6050  IF XH < 0 THEN GOTO 6040
6060  VTAB 23: INPUT "LOWER Y";YL$:YL= VAL (YL$)
6070  IF YL > 0 THEN GOTO 6060
6080  YH=YL+XH-XL: IF YH < 0 THEN GOTO 6060
6100  RETURN
```

The preceeding program can be modified to graph conics using the focus-directrix definition. That is a conic will be drawn if the location of the focus is given, a directrix in the form $Ax+By+C=0$ is provided and the eccentricity is provided.

To make your copy first replace lines 1010 to 1080 with the following.

```
1010  VTAB 21 : PRINT "FOCUS (X, Y) LINE AX+BY+C=0 ECCENTRICTY E"
1020  VTAB 22 : HTAB 1 : INPUT "X=";F$ : F=VAL(F$)
1030  VTAB 22 : HTAB 7 : INPUT "Y=";V$ : V=VAL(V$)
1040  VTAB 22 : HTAB 13 : INPUT "A=";A$ : A=VAL(A$)
1050  VTAB 22 : HTAB 19 : INPUT "B=";B$ : B=VAL(B$)
1060  VTAB 22 : HTAB 25 : INPUT "C=";C$ : C=VAL(C$)
1070  VTAB 22 : HTAB 31 : INPUT "E=";E$ : E=VAL(E$)
1080  A1=E^2×A^2-A^2-B^2
```

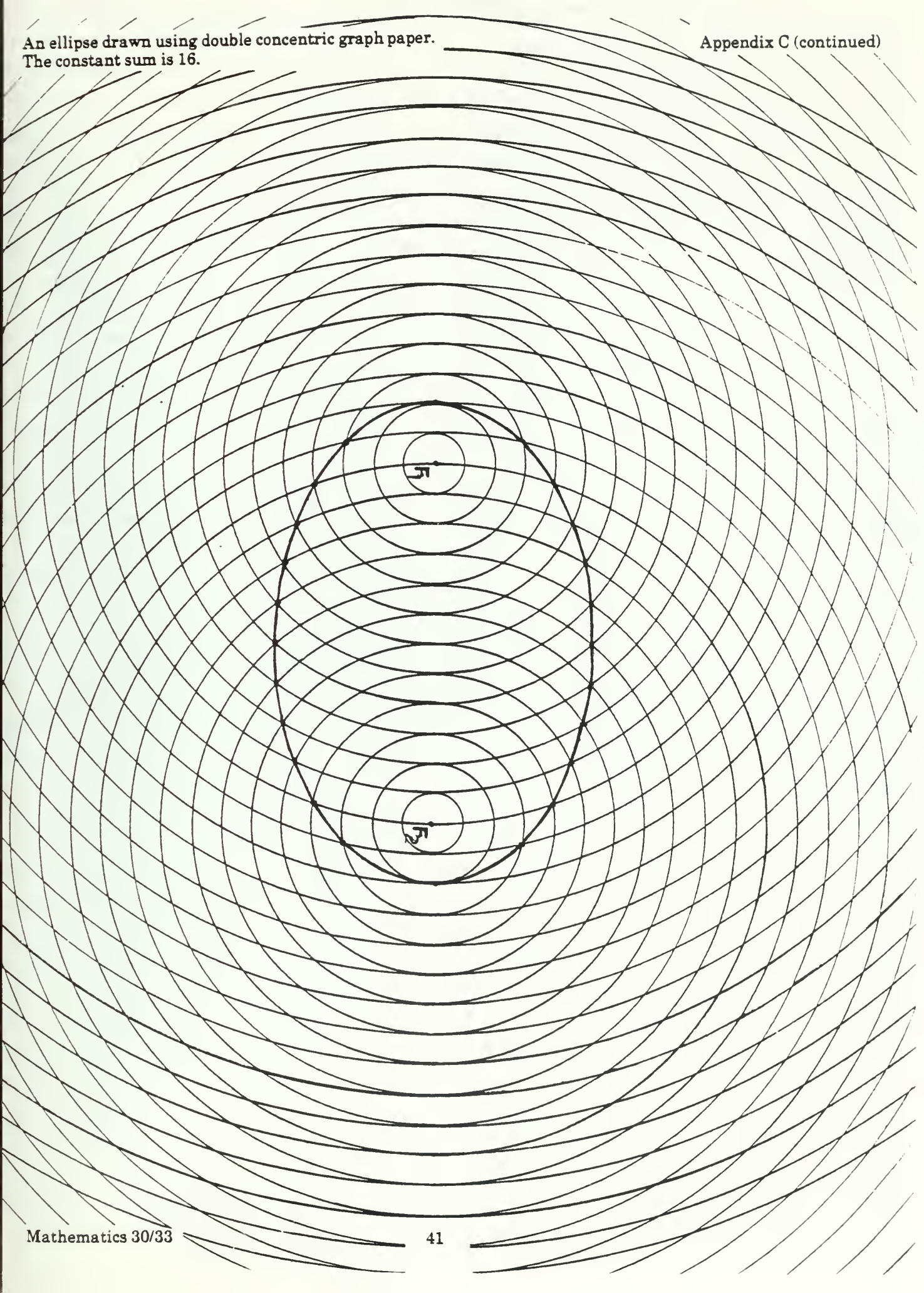
Then include the following lines.

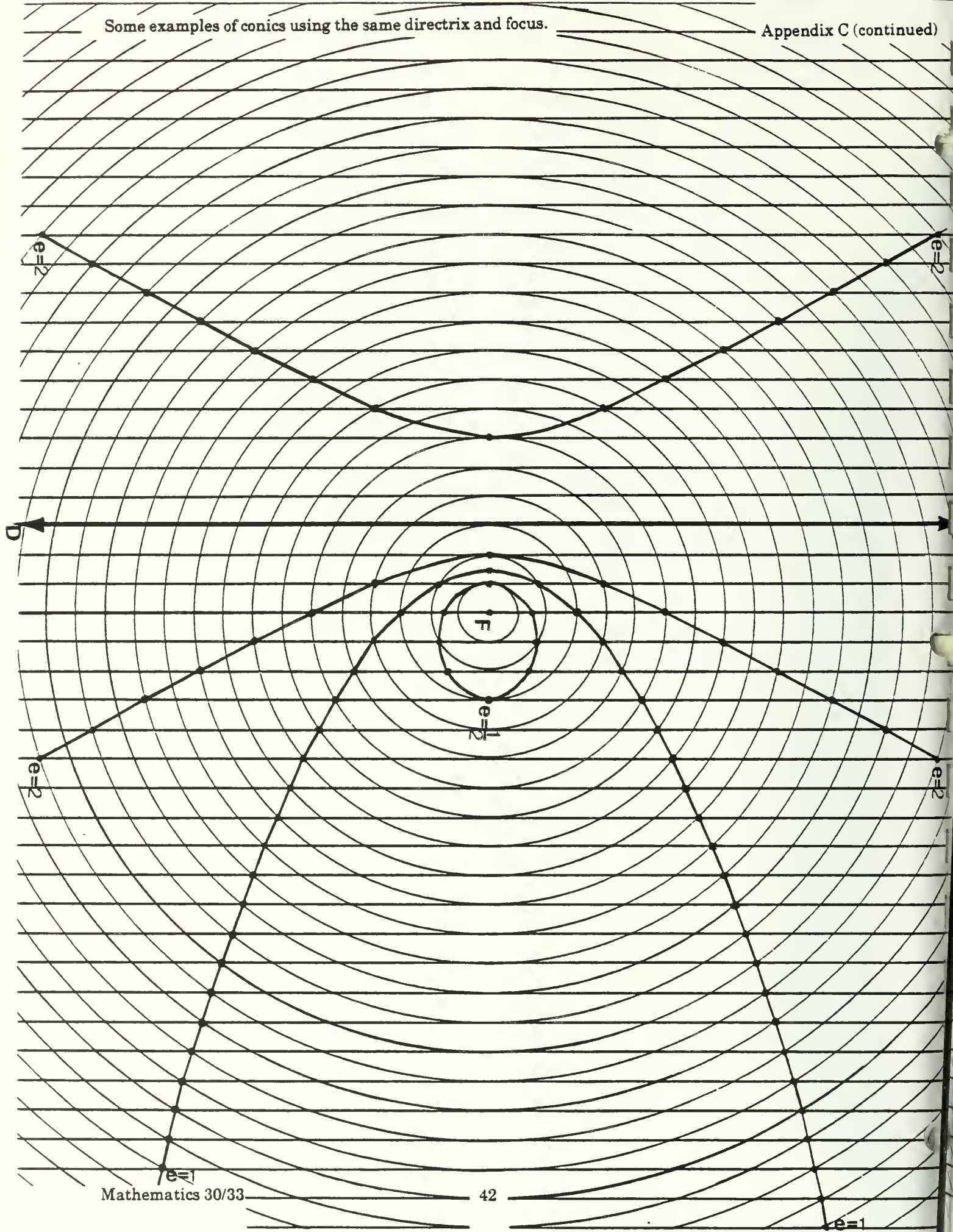
```
1090  B1=2×A×B×E^2
1100  C1=E^2×B^2-A^2-B^2
1110  D1=2×A×C×E^2+2×A^2×F+2×B^2×F
1120  E1=2×B×C×E^2+2×A^2×V+2×B^2×V
1130  F1=E^2×C^2-A^2×F^2-A^2×V^2-B^2×F^2-B^2×V^2
1140  A=A1 : B=B1 : C=C1 : D=D1 : E=E1 : F=F1
1150  RETURN
```

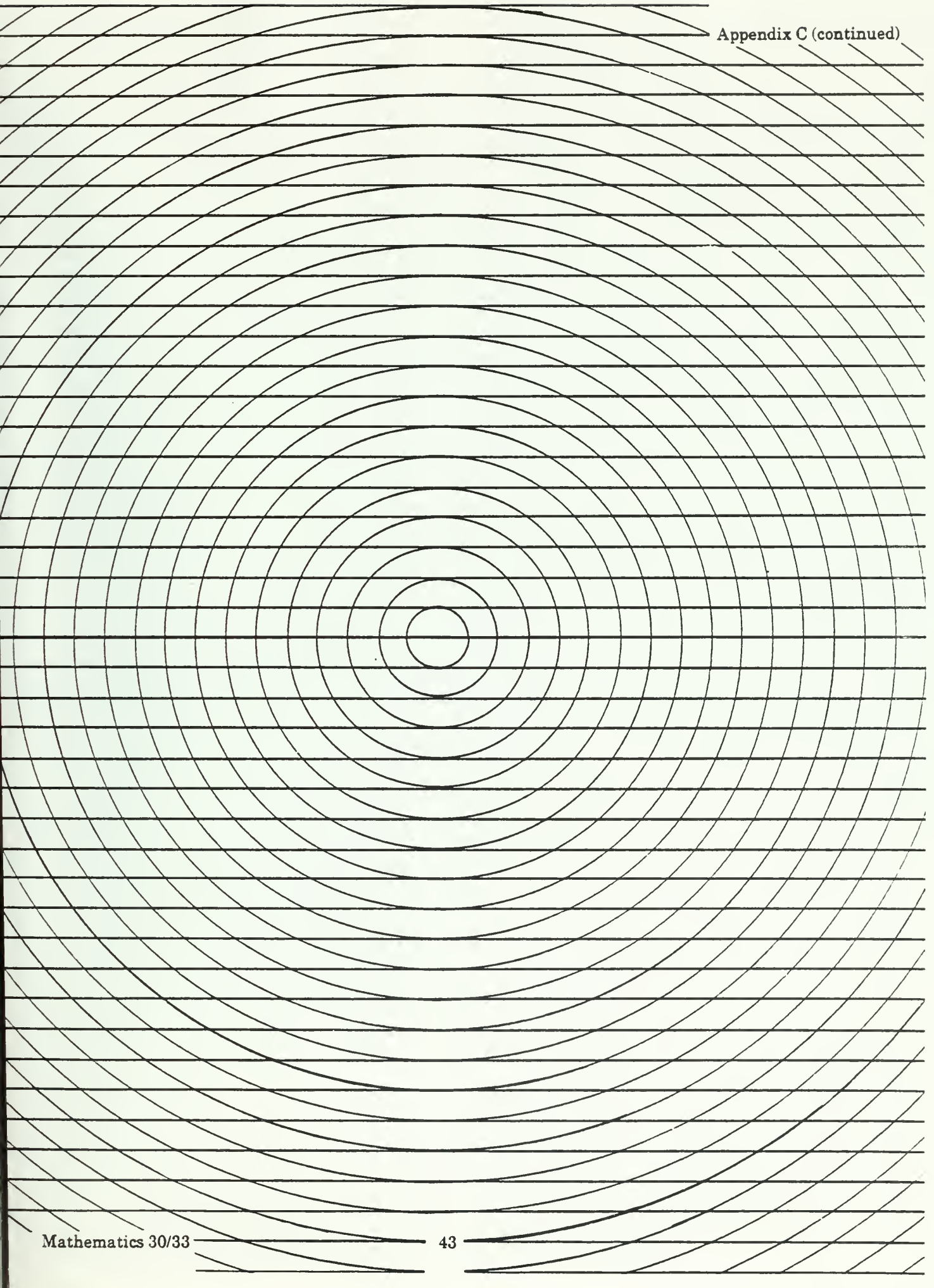
When you have typed these lines, save the program using SAVE FOCUS DIR CONIC.

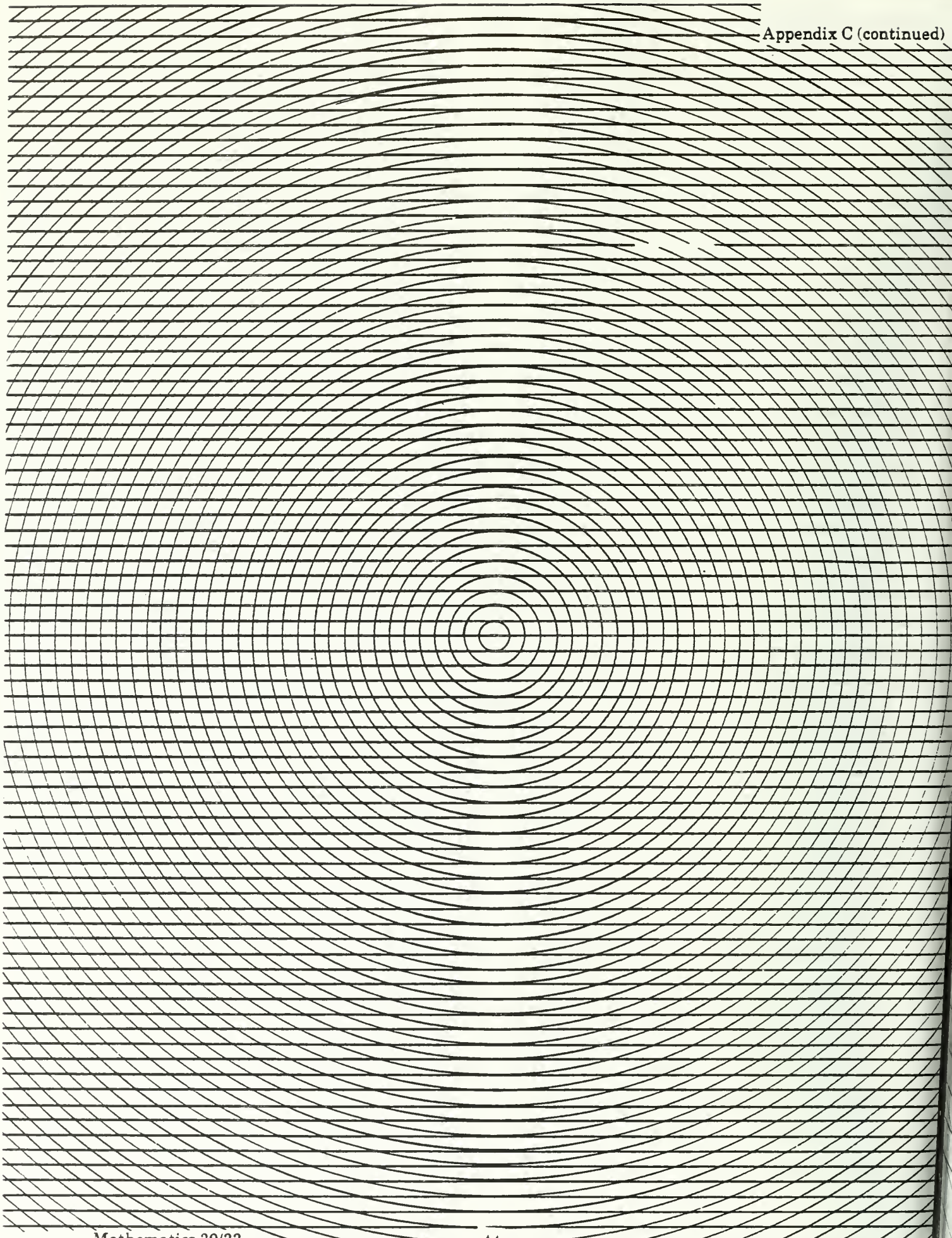
SAMPLE GRAPH PAPERS

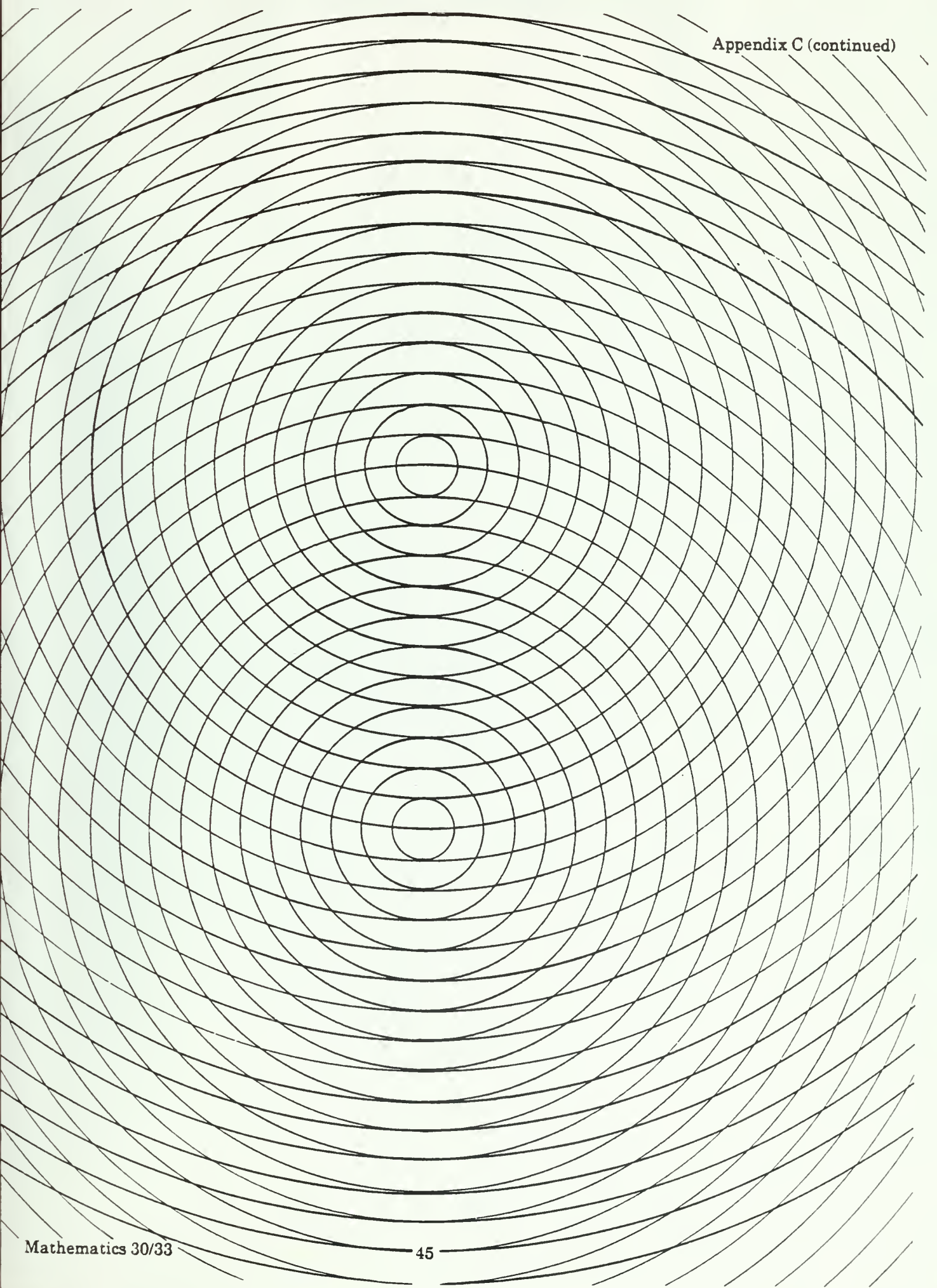
An ellipse drawn using double concentric graph paper.
The constant sum is 16.




e=2
e=2
D
F
e=½
e=2
e=2
e=1
e=1







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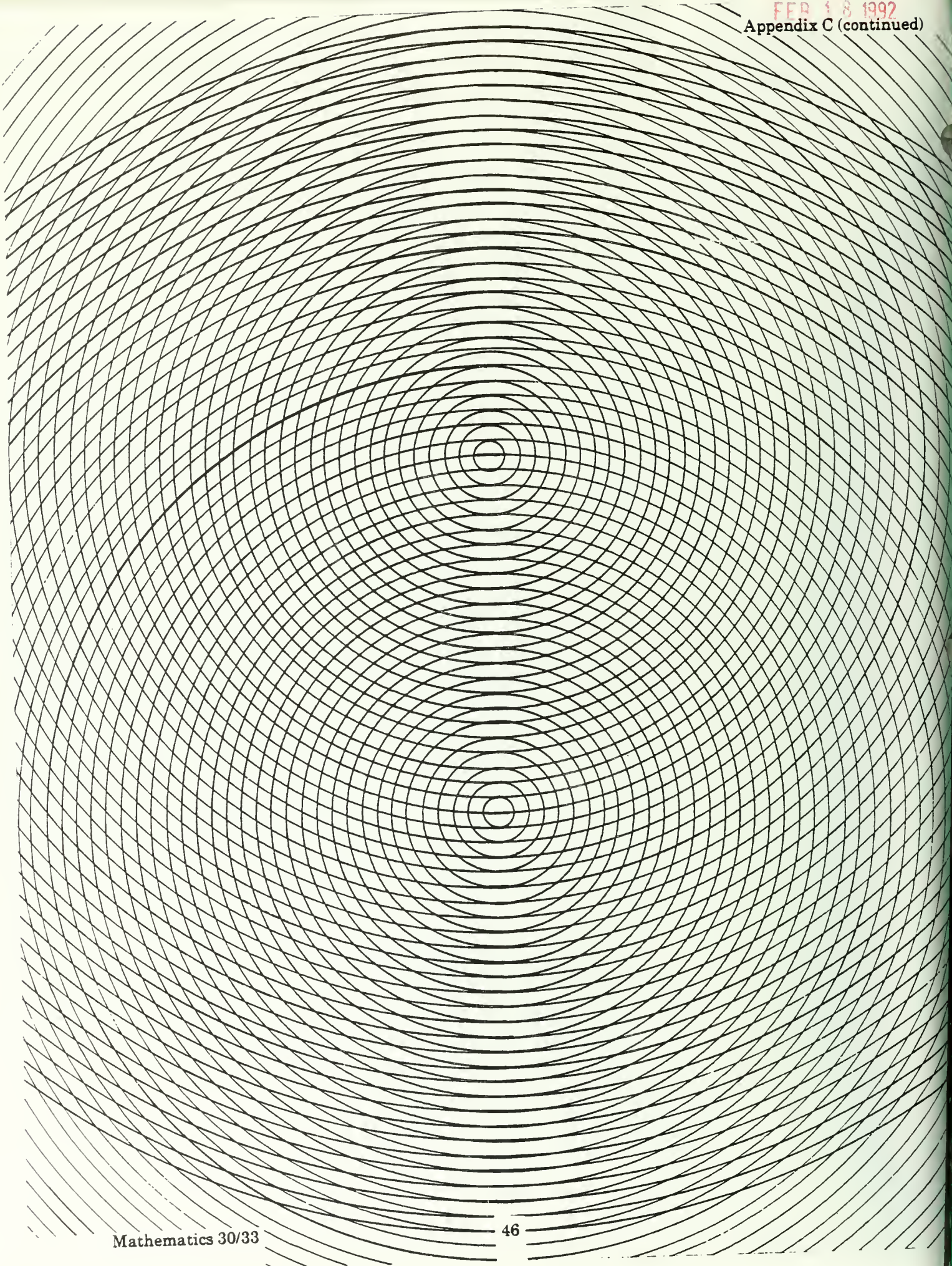

QA 14 C22 A3 A3144 1991
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